

Coordinated Path Following for a Class of Underactuated Multi-agent System using Nested Set Stabilization

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Abstract—This paper addresses the multi-agent coordinated path following problem consisting of a class of underactuated vehicles capable of following a class of closed embedded curves in the three-dimensional space and proposes a solution using a nested set stabilization approach. The stabilization of the first set restricts each agent to the assigned path following manifold and ensures that each system stays on the path indefinitely when initialized in the neighborhood of the path. Stabilizing the second set, contained inside the first, solves the position, velocity, acceleration, or jerk-related coordination constraints. The third set, contained in the second set, satisfies the yaw, or yaw-rate-related coordination constraints. The reduction theorem establishes the asymptotic stability of each nested set, and as a result, the multi-agent coordinated path following problem is solved.

I. INTRODUCTION

Teamwork and cooperation are fundamental requirements for performing complex engineering tasks in several mobile robotic applications [1]. This work considers a multi-agent system consisting of N underactuated vehicles moving in the three-dimensional space. It is assumed that each underactuated vehicle (interchangeably referred to as robot, agent, or system for brevity) is capable of producing torque about each of the three body axis and a thrust force in the direction of one of the body axes [2]. Examples of such robots include quadrotors, under-water vehicles, satellites, and tail-sitting robots [3]. Loosely speaking, a multi-agent coordinated path following problem entails each agent following a given path and satisfying a given coordination constraint while staying on the path.

In the existing literature, the multi-agent coordination and formation problem is divided into several categories based on the problem setup [4]. The problem of achieving a desired formation requires each agent to maintain a suitable position relative to the other robots. When such a formation is time-varying, it leads to the formation tracking problem [5]. The formation tracking problem is generally solved by designing reference trajectories for each agent and forcing each system to follow its respective curves. These two problems can be solved using Lyapunov-based approaches, graph rigidity approaches, receding horizon approaches, and potential-functions-based approaches [4]. A popular method of solving a multi-agent coordination problem is the so-called leader-follower approach [6]. This approach is susceptible to failure as the whole mission fails in the case of the leader's failure. Another way to solve the multi-agent coordination problem is

by using the virtual structure approach [7]. A fundamental shortcoming of all the aforementioned methods is that the agents may leave the path while tracking the desired path parameter. Hence these methods cannot guarantee the path invariance of each agent. Moreover, these methods are more prone to communication failures than our proposed approach.

To overcome the shortcomings mentioned above, we cast the multi-agent coordinated path following problem as a nested set stabilization problem [8], [9]. This approach breaks down the multi-agent path following control design task into a three-stage process. In the first stage, all paths are treated as a set, and this set (called the path following manifold) is made attractive and invariant; this condition implies that all agents converge to the path and stay on the path forever. In the second stage, a second set (called the position-related coordination set) contained in the first set is stabilized relative to the first set, i.e., the path following manifold; by doing so, position or velocity synchronization is achieved. In the third stage, a third set (called the heading-related synchronization set) contained inside the second set is stabilized relative to the second set, i.e., the position-related coordination set; by doing so, the yaw-related synchronization is achieved. The subsequent sections specify conditions under which the multi-agent system enjoys asymptotic stability, given that the nested sets are relatively asymptotic stable.

The problem of coordinated path following using a nested set stabilization has been studied previously for systems consisting of relatively simple dynamical agents, such as the case of N unicycles, a unicycle and a robotic manipulator [9]. However, an extension of this approach to aerial robots is non-trivial. It has not been considered before for systems consisting of multiple underactuated vehicles capable of operating in three-dimensional space. Moreover, we solve a multi-agent coordinated path following problem for a class of three-dimensional underactuated vehicles capable of following a large family of closed embedded curves and performing coordination along these paths.

One of the key advantages of our nested set stabilization approach is that the agents do not need to share any information for the first (path following) stage. Therefore, the agents stay on the path even in the case of complete communication failure. The limitation of the proposed approach is the requirement of the complete connectivity of the graph. Moreover, our proposed approach is centralized and lacks a collision avoidance mechanism.

We have made the following contributions. 1) Novel definitions of position-related and heading-related coordination

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maps; Definition 2.1 and 2.2, respectively. 2) Existence of non-empty controlled invariant sets; Proposition 4.1, 4.2, and 4.4. 3) Sufficient conditions that guarantee a well defined vector relative degree with respect to position-related and heading-related coordination maps; Corollary 4.3 and 4.5, respectively.

All proofs are removed due to space limitations and will appear elsewhere.

A. Notation and mathematical preliminaries

For a point $x \in \mathbb{R}^n$, the Euclidean norm is denoted by $\|x\|$, and the distance of a point x from a subset $S \subset \mathbb{R}^n$ is represented by $\|x\|_S := \inf_{y \in S} \|x - y\|$. A k -dimensional vector x is represented as $\text{col}(x_1, x_2, \dots, x_k) := [x_1, x_2, \dots, x_k]^\top$, where \top denotes transposition. An all-ones (and all-zeroes) vector of dimension n is represented by $\mathbf{1}_N$ (and $\mathbf{0}_N$), respectively. For a full-rank matrix M , its annihilator is denoted by M^\perp . For two maps $h : A \rightarrow B$ and $s : B \rightarrow C$, their composition is denoted by $s \circ h : A \rightarrow C$. The domain of a function σ is a set represented by \mathbb{D} . The domain set for closed curve σ is given as $\mathbb{D} = \mathbb{R} \bmod P$, where P is the length of the curve, also referred to as the period of σ , i.e., for every $\lambda \in \mathbb{D}$, we have $\sigma(\lambda + P) = \sigma(\lambda)$. Given a C^1 map $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and a point $p \in \mathbb{R}^n$, we denote $df_p := \left. \frac{\partial f}{\partial x} \right|_p$. For two smooth maps $f, g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and a smooth real-valued map $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}$, the notation used for the iterative Lie derivatives is as follows: $L_g^0 \lambda := \lambda$, $L_g^k \lambda := L_g(L_g^{k-1} \lambda)$, $L_g L_f \lambda := L_g(L_f \lambda)$. We use the following notation to represent trigonometric functions; $S_i := \sin(i)$, $C_i := \cos(i)$, $T_i := \tan(i)$ for $i \in \{\phi, \theta, \psi\}$. Let $\mathcal{X} \subset \mathbb{R}^n$, $x \in \mathcal{X}$, $f : \mathcal{X} \rightarrow \mathcal{X}$ be locally Lipschitz on \mathcal{X} and the dynamical system is given by $\Sigma : \dot{x} = f(x)$. For details about the notion of set stability, relative set stability, and reduction theorem, the readers are referred to [10]. The reduction theorem, an essential tool for establishing asymptotic set stability, is reproduced here for convenience.

Theorem 1.1 (Reduction theorem): Let $\Gamma_1 \subset \Gamma_2 \subset \mathcal{X}$ be closed positively invariant sets for Σ , and assume that Γ_1 is compact. If Γ_1 is asymptotically stable relative to Γ_2 and Γ_2 is asymptotically stable, then Γ_1 is asymptotically stable.

II. SYSTEM MODEL AND PROBLEM FORMULATION

This section proposes the standard mathematical model of a single agent and a coordinated path following problem for a multi-agent system, i.e., N underactuated systems [11]. We consider this coordination task under the assumption that the individual agents can mutually communicate the required state information without any delay or interference. We further assume that the undirected connectivity graph constituted by the N agents of the system is complete, i.e., all the agents can mutually exchange the minimal information needed for the coordination task. Let the position and velocity of the single robot is given by $\chi \in \mathbb{R}^3$ and $\dot{\chi} \in \mathbb{R}^3$, respectively. The Euler angles (roll-pitch-yaw) are denoted by $\Phi = \text{col}(\phi, \theta, \psi) \in \mathbb{R}^3$ and body rates are denoted by $\Omega = \text{col}(p, q, r) \in \mathbb{R}^3$. Let $m \in \mathbb{R}$ be the mass of the system, $g \in \mathbb{R}$ is the acceleration due to gravity,

and $b_3 = \text{col}(0, 0, 1)$ is the unit vector in z -direction. Let $J := \text{diag}(J_x, J_y, J_z) \in \mathbb{R}^{3 \times 3}$ represent the inertia of the vehicle and $\tau := \text{col}(\tau_p, \tau_q, \tau_r) \in \mathbb{R}^3$ be the control torque. The thrust control input $u_t \in \mathbb{R}$ is assumed to be non-zero throughout the mission. The dynamics of a the n^{th} underactuated agent¹ can be locally represented as

$$\begin{aligned} \dot{\Phi}^n &= M^n \Omega^n \\ J \dot{\Omega}^n &= \tau^n - (\Omega^n \times J \Omega^n) \\ \dot{\chi}^n &= v^n \\ \dot{v}^n &= m g b_3 - u_t^n R^n b_3, \end{aligned} \quad (1)$$

where M^n is the directional cosine matrix and the rotation matrix R^n can be represented in terms of Euler angles, for details see [11]. The state vector of the n^{th} agent is defined as $\mathbf{x}^n := \text{col}(\Phi^n, \Omega^n, \chi^n, \dot{\chi}^n) \in \mathbb{R}^{12}$ and the inputs are $\tau_p^n, \tau_q^n, \tau_r^n$, and u_t^n .

A. Extended model of multi-agent system

As shown in [11], [12], dynamic extension is required to achieve path following via transverse feedback linearization. A well defined vector relative degree is achieved by delaying the appearance of the control input u_t^n with the help of two integrators. Let $x_{13}^n := u_t^n$, $x_{14}^n := \dot{u}_t^n$, and $\dot{x}_{14}^n := u_d^n$. The model of the n^{th} agent (1) after dynamic extension gets the following form:

$$\begin{aligned} \dot{\Phi}^n &= M \Omega^n \\ J \dot{\Omega}^n &= \tau^n - (\Omega^n \times J \Omega^n) \\ \dot{\chi}^n &= v^n \\ \dot{v}^n &= m g b_3 - u_t^n R^n b_3 \\ \dot{x}_{13}^n &= x_{14}^n \\ \dot{x}_{14}^n &= u_d^n, \end{aligned} \quad (2)$$

for $n \in \{1, \dots, N\}$. With a slight abuse of notation, the extended state of the n^{th} agent is redefined as $\mathbf{x}^n := \text{col}(x_1^n, \dots, x_{14}^n) \in \mathbb{R}^{14}$, and, for some appropriate maps f and g , (2) can be compactly written as

$$\dot{\mathbf{x}}^n = f(\mathbf{x}^n) + g(\mathbf{x}^n) \mathbf{u}^n,$$

where $\mathbf{u}^n := \text{col}(\tau_p^n, \tau_q^n, \tau_r^n, u_d^n) \in \mathbb{R}^4$. The output of the n^{th} system is the position of the center of gravity of the agent in the inertial frame

$$y^n = h^n(\mathbf{x}^n) = [x_7^n \quad x_8^n \quad x_9^n]^\top, \quad (3)$$

where $h^n : \mathbb{R}^{14} \rightarrow \mathbb{R}^3$. We define the combined state of the multi-agent system as $\mathbf{x} := \text{col}(\mathbf{x}^1, \dots, \mathbf{x}^N) \in \mathbb{R}^{14N}$, and $\mathbf{u} := \text{col}(\mathbf{u}^1, \dots, \mathbf{u}^N) \in \mathbb{R}^{4N}$.

B. Multi-agent Path following manifold

Consider a closed C^∞ curve $\gamma^n : \mathbb{D} \rightarrow \mathbb{R}^3$, $\mathbb{D} \ni \lambda^n \mapsto \gamma^n(\lambda) \subset \mathbb{R}^3$ in the n^{th} system's output space. Let

$$\begin{aligned} \sigma^n &: \mathbb{D} \rightarrow \mathbb{R}^3 \\ \lambda &\mapsto \text{col}(\sigma_1^n(\lambda^n), \sigma_2^n(\lambda^n), \sigma_3^n(\lambda^n)) \end{aligned}$$

¹the superscript represents the agent number

be a parameterization of the curve γ^n , and let the curve be regular, i.e., $\|\sigma^n(\lambda^n)\| \neq 0$. As σ^n is a regular curve, we assume, without loss of generality, that it can be unit-speed parameterized, i.e., $\|\sigma^n(\lambda^n)\| = 1$, using arc length parameterization. Similar to [12], the following assumption is made:

Assumption 1: The curve $\gamma^n \subset \mathbb{R}^3$ is a compact embedded sub-manifold of dimension one. For an open set W , a smooth map $s^n : W \subset \mathbb{R}^3 \rightarrow \mathbb{R}^2$ can be defined such that $\gamma^n = (s^n)^{-1}(0)$ with $\text{rank}(ds_y^n) = 2, \forall y^n \in \gamma^n$. Let $\alpha_1^n := s_1^n \circ h^n$ and $\alpha_2^n := s_2^n \circ h^n$ be smooth maps. We lift the smooth path $\gamma^n : \mathbb{D} \rightarrow \mathbb{R}^3$ to the state space of an underactuated system

$$\Gamma^n := \{\mathbf{x}^n \in \mathbb{R}^{14} : \alpha_1^n(\mathbf{x}^n) = \alpha_2^n(\mathbf{x}^n) = 0\}.$$

Making the output y^n of the n^{th} system converge to the path γ^n is equivalent to forcing the states of the system \mathbf{x}^n to converge to the lifted path Γ^n . To asymptotically stabilize Γ^n we find the largest controlled-invariant subset of Γ^n and call it the path following manifold [12]

$$(\Gamma^n)^* = \{\mathbf{x}^n \in \mathbb{R}^{14} : L_f^i \alpha(x) = 0, i = 0, 1, 2, 3, 4\}. \quad (4)$$

By definition, $(\Gamma^n)^* \subset \Gamma^n$ and is invariant. Moreover, it is shown in [12] that $(\Gamma^n)^*$ is a non-empty and compact embedded submanifold. We define the lifted path for all the multi-agent systems as a product of individual lifted paths, i.e., $\Gamma := \Gamma^1 \times \dots \times \Gamma^N$. Similarly, we define the path following manifold for the overall multi-agent system as a product manifold of the individual path following manifolds, i.e., $\Gamma^* := (\Gamma^1)^* \times \dots \times (\Gamma^N)^*$. Clearly, $\Gamma^* \subset \Gamma$ and is compact. We call Γ^* the multi-agent path following manifold.

C. Coordination manifolds

Intuitively, stabilizing the multi-agent path following manifold Γ^* implies that each agent converges to its assigned path and stays on the path for all future times. Once each agent converges “sufficiently close” to the multi-agent path following manifold, we perform coordination. A multi-agent system, consisting of multiple underactuated vehicles, can achieve two types of coordination, i.e., position (or its derivatives) coordination and headings (and its derivative) coordination. We denote the coordination constraints of the former kind by β_η and the latter kind by β_μ .

Definition 2.1: A position-related coordination map for a multi-agent underactuated system is a smooth map $\bar{\beta}_\eta : \Gamma^* \rightarrow \mathbb{R}^c, 0 \leq c \leq \dim(\Gamma^*)$. For a coordinate chart (\mathcal{O}, Ψ_η) , the local representation of the position-related coordination map is given by the following commutative diagram,

$$\begin{array}{ccc} \mathcal{O} \subseteq \Gamma^* & \xrightarrow{\bar{\beta}_\eta} & \mathbb{R}^c \\ \Psi_\eta \downarrow & \nearrow \beta_\eta & \\ \Psi_\eta(\mathcal{O}) \subseteq \mathbb{R}^{\dim(\Gamma^*)} & & \end{array}$$

and $d\beta_\eta$ has rank c everywhere on the local chart (\mathcal{O}, Ψ_η) .

We assume that \mathcal{O} is closed, and since Γ^* is compact, it follows that \mathcal{O} is also a compact set. The above definition allows us to define a position-related coordination set $\mathcal{P} = \bar{\beta}_\eta^{-1}(0)$. Informally, stabilizing the set \mathcal{P} implies achieving position-related coordination. Moreover, we require that if the agents have initially satisfied position-related coordinated constraints, they remain coordinated for all future times. In other words, this requires stabilizing the largest controlled-invariant subset of \mathcal{P} . We denote this invariant set by \mathcal{P}^* , which is characterized in the subsequent sections.

Definition 2.2: A heading-related coordination map for a multi-agent underactuated system is a smooth map $\bar{\beta}_\mu : \mathcal{P}^* \rightarrow \mathbb{R}^d, 0 \leq d \leq \dim(\mathcal{P}^*)$. For a coordinate chart (\mathcal{O}, Ψ_μ) the local representation of the position-related coordination map is given by the following commutative diagram

$$\begin{array}{ccc} \mathcal{O} \subseteq \mathcal{P}^* & \xrightarrow{\bar{\beta}_\mu} & \mathbb{R}^d \\ \Psi_\mu \downarrow & \nearrow \beta_\mu & \\ \Psi_\mu(\mathcal{O}) \subseteq \mathbb{R}^{\dim(\mathcal{P}^*)} & & \end{array}$$

and $d\beta_\mu$ has rank d everywhere on the local chart (\mathcal{O}, Ψ_μ) . Similar to the previous definition, it follows that $\mathcal{O} \subseteq \mathcal{P}^*$ is compact, given that \mathcal{O} is closed. The above definition allows us to define a heading-related coordination set $\mathcal{H} = \beta_\mu^{-1}(0)$.

D. Problem statement

Given N underactuated systems (2), and N paths satisfying Assumption 1, we seek a dynamic feedback controller $\mathbf{u} : \mathbb{R}^{14N} \rightarrow \mathbb{R}^{4N}$ such that the resulting closed-loop multi-agent system satisfies the following objectives:

- PF** the path following manifold Γ^* is locally asymptotically stable;
- PC** the position-related synchronization manifold \mathcal{P}^* is locally asymptotically stable relative to Γ^* ;
- HC** the heading-related synchronization manifold \mathcal{H}^* is locally asymptotically stable relative to \mathcal{P}^* ;
- AS** moreover, both \mathcal{P}^* and \mathcal{H}^* are locally asymptotically stable.

We have formulated the problem in a hierarchical structure consisting of three-layered objectives. The first objective **PF** has the highest priority. Intuitively, by achieving **PF**, each agent converges to its assigned path and stays on the path indefinitely. Note that to achieve **PF** and to maintain it requires no information exchange among the agents. Once the agents are on the path-following manifold, in the next phase, they attain their position-related objectives **PC** and subsequently, the heading-related objectives **HC** through mutual exchange of information. Note that **PC** takes precedence over **HC**. Thus, we cast the coordinated path following problem as a three-staged nested set stabilization problem. First we stabilize Γ^* , followed by \mathcal{P}^* , and finally the system converges to \mathcal{H}^* .

III. PATH FOLLOWING

To achieve **PF**, we convert the system into an exact linear form by performing transverse feedback linearization of each

agent [12]. To achieve this, a mapping is introduced that associates to a point y^n in the output space of the n^{th} agent, sufficiently close to the path, a number in the domain \mathbb{D} that minimizes the distance from the path γ^n . Let $\gamma_\epsilon^n \subset \mathbb{R}^3$ be a tubular neighbourhood of the path γ^n and define the map

$$\begin{aligned} \varpi^n : \gamma_\epsilon^n &\rightarrow \mathbb{D} \\ y^n &\mapsto \arg \inf_{\lambda^n \in \mathbb{D}} \|y^n - \sigma^n(\lambda^n)\|. \end{aligned} \quad (5)$$

The above function² is smooth provided that γ_ϵ^n is a sufficiently small “tube” around the curve γ^n . Let $\tilde{h}^n : \mathbb{R}^{14} \rightarrow \mathbb{R}^4$ and $\tilde{s}^n : \mathbb{R}^4 \rightarrow \mathbb{R}$ such that $d_{x_3} \tilde{h}^n \neq 0$. With these definitions and Assumption 1, we define the “virtual output” function \hat{y}^n to be

$$\hat{y}^n = \begin{bmatrix} \alpha_1^n(x_7^n, x_8^n, x_9^n) \\ \alpha_2^n(x_7^n, x_8^n, x_9^n) \\ \pi_1^n(x_7^n, x_8^n, x_9^n) \\ \pi_2^n(x_3^n, x_7^n, x_8^n, x_9^n) \end{bmatrix} = \begin{bmatrix} s_1^n \circ h^n \\ s_2^n \circ h^n \\ \varpi^n \circ h^n \\ \tilde{s}^n \circ \tilde{h}^n \end{bmatrix}. \quad (6)$$

The following result establishes that the virtual output function \hat{y}^n is a flat output for an agent.

Lemma 3.1: For n^{th} agent with model (2), the output (6) yields a well-defined vector relative degree of $\{4, 4, 4, 2\}$ at each point on $(\Gamma^n)^* \cap \{\mathbf{x}^n \in \mathbb{R}^{14} : x_{13}^n \neq 0, x_1^n \neq \pm 90^\circ, x_2^n \neq \pm 90^\circ\}$.

The above result allows us to define a coordinate transformation for each agent [12].

Corollary 3.2: Let $x^* \in (\Gamma^n)^* \cap \{\mathbf{x}^n \in \mathbb{R}^{14} : x_1^n \neq \pm 90^\circ, x_2^n \neq \pm 90^\circ, x_{13}^n \neq 0\}$. There exists a neighbourhood $U \subset \mathbb{R}^{14}$ containing x^* such that the mapping $T : U \subset \mathbb{R}^{14} \rightarrow T(U) \subset \mathbb{R}^{14}$, defined by

$$T(\mathbf{x}^n) = \begin{bmatrix} L_f^{j-1} \alpha_1^n(\mathbf{x}^n) \\ L_f^{j-1} \alpha_2^n(\mathbf{x}^n) \\ L_f^{j-1} \pi_1^n(\mathbf{x}^n) \\ L_f^{k-1} \pi_2^n(\mathbf{x}^n) \end{bmatrix} := \begin{bmatrix} \xi_j^n \\ \zeta_j^n \\ \eta_j^n \\ \mu_k^n \end{bmatrix}, \quad (7)$$

is a diffeomorphism for $j \in \{1, 2, 3, 4\}$ and $k \in \{1, 2\}$.

Thus in a neighbourhood of x^* , by an appropriate feedback transformation, similar to the one in [12], each agent is simply reduced to four decoupled chains of integrators:

$$\begin{aligned} \dot{\xi}_1^n &= \xi_2^n & \dot{\zeta}_1^n &= \zeta_2^n & \dot{\eta}_1^n &= \eta_2^n & \dot{\mu}_1^n &= \mu_2^n \\ \dot{\xi}_2^n &= \xi_3^n & \dot{\zeta}_2^n &= \zeta_3^n & \dot{\eta}_2^n &= \eta_3^n & \dot{\mu}_2^n &= v_\mu^n \\ \dot{\xi}_3^n &= \xi_4^n & \dot{\zeta}_3^n &= \zeta_4^n & \dot{\eta}_3^n &= \eta_4^n & & \\ \dot{\xi}_4^n &= v_\xi^n & \dot{\zeta}_4^n &= v_\zeta^n & \dot{\eta}_4^n &= v_\eta^n & & \end{aligned} \quad (8)$$

$\underbrace{\xi\text{-subsystem}} \quad \underbrace{\zeta\text{-subsystem}} \quad \underbrace{\eta\text{-subsystem}} \quad \underbrace{\mu\text{-subsystem}}$

The virtual output (6) is a flat output of each agent (2) because these outputs transform the system to a fully linear system. A linear control law, similar to [12], is designed

²For a large class of functions of the form $y = f(x)$ or $x = f(y)$, a more efficient choice for ϖ^n is pr_i , where pr_i is natural projection of the i -th component.

which provides exponential stability of the origin of ξ - and ζ -subsystems

$$v_\xi^n = - \sum_{i=1}^4 k_{\xi_i}^n \xi_i^n, \quad v_\zeta^n = - \sum_{i=1}^4 k_{\zeta_i}^n \zeta_i^n, \quad (9)$$

for some positive gains $k_{\xi_i}^n$ and $k_{\zeta_i}^n$ for $i \in \{1, \dots, 4\}$, and for $n \in \{1, \dots, N\}$. The control law (9) forces the output $y(\mathbf{x}^n)$ to the corresponding path γ^n , which makes all the states of each agent converge to their respective path following manifold $(\Gamma^n)^*$, which can be expressed as

$$(\Gamma^n)^* = \{\mathbf{x}^n \in \mathbb{R}^{14} : (\xi_i^n, \zeta_i^n) = 0, \forall 1 \leq i \leq 4\}.$$

We characterizes the multi-agent path following manifold Γ^* , i.e., $\Gamma^* := (\Gamma^1)^* \times \dots \times (\Gamma^N)^*$. Consequently, the states of the multi-agent system converge to Γ^* and stay in the set for all future times. In other words, the first specification of the problem, i.e., **PF** is achieved.

IV. COORDINATION

In this section, we present dynamics of the multi-agent system of N underactuated systems when restricted to the corresponding path of each agent. On the path following manifold Γ^* , the states of ξ -subsystem and ζ -subsystem are all zero, and the feedback linearized dynamics (8) are reduced to two parts, i.e.,

- η -subsystem: governs position, velocity, acceleration, and jerk along the path, and
- μ -subsystem: governs the heading and rate of change of heading of the system.

Let $\boldsymbol{\eta}_1 := \text{col}(\eta_1^1, \dots, \eta_1^N)$, $\boldsymbol{\eta}_2 := \text{col}(\eta_2^1, \dots, \eta_2^N)$, $\boldsymbol{\eta}_3 := \text{col}(\eta_3^1, \dots, \eta_3^N)$, and $\boldsymbol{\eta}_4 := \text{col}(\eta_4^1, \dots, \eta_4^N)$, and $\mathbf{v}_\eta := \text{col}(v_\eta^1, \dots, v_\eta^N)$. Next we stack η -states of all the N agents and define $\boldsymbol{\eta} := \text{col}(\boldsymbol{\eta}_1, \dots, \boldsymbol{\eta}_4)$. This allows us to write the $\boldsymbol{\eta}$ -dynamics of the multi-agent system when restricted to the set Γ^* as

$$\dot{\boldsymbol{\eta}} = \begin{bmatrix} 0_{N \times N} & I_{N \times N} & 0_{N \times N} & 0_{N \times N} \\ 0_{N \times N} & 0_{N \times N} & I_{N \times N} & 0_{N \times N} \\ 0_{N \times N} & 0_{N \times N} & 0_{N \times N} & I_{N \times N} \\ 0_{N \times N} & 0_{N \times N} & 0_{N \times N} & 0_{N \times N} \end{bmatrix} \boldsymbol{\eta} + \begin{bmatrix} 0_{N \times N} \\ 0_{N \times N} \\ 0_{N \times N} \\ I_{N \times N} \end{bmatrix} \mathbf{v}_\eta. \quad (10)$$

Let $\boldsymbol{\mu}_1 := \text{col}(\mu_1^1, \dots, \mu_1^N)$, $\boldsymbol{\mu}_2 = \text{col}(\mu_2^1, \dots, \mu_2^N)$, $\boldsymbol{\mu} = \text{col}(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2)$, and $\mathbf{v}_\mu := \text{col}(v_\mu^1, \dots, v_\mu^N)$. Then the $\boldsymbol{\mu}$ dynamics on Γ^* can be represented in the following linear form:

$$\dot{\boldsymbol{\mu}} = \begin{bmatrix} 0_{N \times N} & I_{N \times N} \\ 0_{N \times N} & 0_{N \times N} \end{bmatrix} \boldsymbol{\mu} + \begin{bmatrix} 0_{N \times N} \\ I_{N \times N} \end{bmatrix} \mathbf{v}_\mu. \quad (11)$$

A. η -coordination

To perform coordination among η states of each agent, we consider the coordination map $\beta_\eta : \Psi_\eta(\mathcal{O}) \subseteq \mathbb{R}^{\dim(\Gamma^*)} \rightarrow \mathbb{R}^c$ given in Definition 2.1. This allows us to define the η -coordination set

$$\mathcal{P} := \{(\boldsymbol{\eta}, \boldsymbol{\mu}) \in \Gamma^* : \beta_\eta(\boldsymbol{\eta}) = 0\}.$$

We represent the above set with the letter \mathcal{P} as we use this set to synchronize position and the higher derivatives of position of the multi-agent system. It should be noted that $\beta_\eta : \Gamma^* \rightarrow \mathbb{R}^c$ may not be defined globally because in general Γ^* has a structure of a smooth manifold. However, one could use the local parameterization of the map β_η . To satisfy the general coordination condition $\beta_\eta(\boldsymbol{\eta}) = 0$, there may or may not exist a suitable coordination controller, and at times this leads to practically non-feasible coordination scenarios. Therefore, we restrict the subsequent discussion to the following special cases of the map β_η .

Proposition 4.1: If $(\boldsymbol{\eta}, \boldsymbol{\mu}) \in \mathcal{P}$, $\partial_{\boldsymbol{\eta}_2}\beta_\eta = \partial_{\boldsymbol{\eta}_3}\beta_\eta = \partial_{\boldsymbol{\eta}_4}\beta_\eta = 0$, and

$$\begin{aligned} \text{Im}(\partial_{\boldsymbol{\eta}_1}(\beta_\eta)) &\ni \partial_{\boldsymbol{\eta}_1}[\partial_{\boldsymbol{\eta}_1}\{\partial_{\boldsymbol{\eta}_1}(\beta_\eta)\boldsymbol{\eta}_2\}\boldsymbol{\eta}_2 \\ &+ \partial_{\boldsymbol{\eta}_1}(\beta_\eta)\boldsymbol{\eta}_3]\boldsymbol{\eta}_2 \\ &+ \partial_{\boldsymbol{\eta}_2}\{\partial_{\boldsymbol{\eta}_1}(\beta_\eta)\boldsymbol{\eta}_2\}\boldsymbol{\eta}_2 + \partial_{\boldsymbol{\eta}_1}(\beta_\eta)\boldsymbol{\eta}_3]\boldsymbol{\eta}_3 \\ &+ \partial_{\boldsymbol{\eta}_1}(\beta_\eta)\boldsymbol{\eta}_4]\boldsymbol{\eta}_2 \\ &+ \partial_{\boldsymbol{\eta}_2}[\partial_{\boldsymbol{\eta}_1}\{\partial_{\boldsymbol{\eta}_1}(\beta_\eta)\boldsymbol{\eta}_2\}\boldsymbol{\eta}_2 + \partial_{\boldsymbol{\eta}_1}(\beta_\eta)\boldsymbol{\eta}_3]\boldsymbol{\eta}_2 \\ &+ \partial_{\boldsymbol{\eta}_2}\{\partial_{\boldsymbol{\eta}_1}(\beta_\eta)\boldsymbol{\eta}_2\}\boldsymbol{\eta}_2 + \partial_{\boldsymbol{\eta}_1}(\beta_\eta)\boldsymbol{\eta}_3]\boldsymbol{\eta}_3 \\ &+ \partial_{\boldsymbol{\eta}_1}(\beta_\eta)\boldsymbol{\eta}_4]\boldsymbol{\eta}_3 \\ &+ \partial_{\boldsymbol{\eta}_3}[\partial_{\boldsymbol{\eta}_1}\{\partial_{\boldsymbol{\eta}_1}(\beta_\eta)\boldsymbol{\eta}_2\}\boldsymbol{\eta}_2 + \partial_{\boldsymbol{\eta}_1}(\beta_\eta)\boldsymbol{\eta}_3]\boldsymbol{\eta}_2 \\ &+ \partial_{\boldsymbol{\eta}_2}\{\partial_{\boldsymbol{\eta}_1}(\beta_\eta)\boldsymbol{\eta}_2\}\boldsymbol{\eta}_2 + \partial_{\boldsymbol{\eta}_1}(\beta_\eta)\boldsymbol{\eta}_3]\boldsymbol{\eta}_3 \\ &+ \partial_{\boldsymbol{\eta}_1}(\beta_\eta)\boldsymbol{\eta}_4]\boldsymbol{\eta}_4, \end{aligned}$$

then, there exists a nonempty controlled-invariant set

$$\mathcal{P}^* = \left\{ (\boldsymbol{\eta}, \boldsymbol{\mu}) \in \mathcal{P} : \beta_\eta = \dot{\beta}_\eta = \ddot{\beta}_\eta = \ddot{\beta}_\eta = 0 \right\}.$$

Proposition 4.2: For $(\boldsymbol{\eta}, \boldsymbol{\mu}) \in \mathcal{P}$, there exists a nonempty controlled-invariant set \mathcal{P}^* nested in Γ^* if either of the following three conditions hold:

- 1) **Velocity coordination:** $\partial_{\boldsymbol{\eta}_1}\beta_\eta = \partial_{\boldsymbol{\eta}_3}\beta_\eta = \partial_{\boldsymbol{\eta}_4}\beta_\eta = 0$, and

$$\begin{aligned} \text{Im}(\partial_{\boldsymbol{\eta}_2}(\beta_\eta)) &\ni \partial_{\boldsymbol{\eta}_2}(\partial_{\boldsymbol{\eta}_2}(\beta_\eta)\boldsymbol{\eta}_3)\boldsymbol{\eta}_3 \\ &+ \partial_{\boldsymbol{\eta}_2}(\beta_\eta)\boldsymbol{\eta}_4]\boldsymbol{\eta}_3 \\ &+ \partial_{\boldsymbol{\eta}_3}(\partial_{\boldsymbol{\eta}_2}(\beta_\eta)\boldsymbol{\eta}_3)\boldsymbol{\eta}_3)\boldsymbol{\eta}_4. \end{aligned} \quad (12)$$

- 2) **Acceleration coordination:** $\partial_{\boldsymbol{\eta}_1}\beta_\eta = \partial_{\boldsymbol{\eta}_2}\beta_\eta = \partial_{\boldsymbol{\eta}_4}\beta_\eta = 0$, and

$$\text{Im}(\partial_{\boldsymbol{\eta}_3}\beta_\eta) \ni \partial_{\boldsymbol{\eta}_3}(\partial_{\boldsymbol{\eta}_3}\beta_\eta)\boldsymbol{\eta}_4. \quad (13)$$

- 3) **Jerk coordination:** $\partial_{\boldsymbol{\eta}_1}\beta_\eta = \partial_{\boldsymbol{\eta}_2}\beta_\eta = \partial_{\boldsymbol{\eta}_3}\beta_\eta = 0$.

Corollary 4.3: For the coordination function $\beta_\eta : \Psi_\eta(\mathcal{O}) \subseteq \mathbb{R}^{\dim(\Gamma^*)} \rightarrow \mathbb{R}^c$ and a fixed $i \in \{1, 2, 3, 4\}$, assuming that $\text{rank} \partial_{\boldsymbol{\eta}_i}\beta_\eta = c$, and $\partial_{\boldsymbol{\eta}_k}\beta_\eta = 0$, for $k \in \{1, 2, 3, 4\} \setminus i$, the vector relative degree of the $\boldsymbol{\eta}$ -subsystem (10) with respect to the function β_η is given by $(5-i)\mathbf{1}_N$.

Given a position-related constraint function β_η , with corresponding vector-relative degree $r\mathbf{1}_N$, Corollary 4.3 allows us to express its r^{th} derivative as follows:

$$\beta_\eta^{(r)} = \sum_{j=1}^{r-1} \partial_{\boldsymbol{\eta}_j}\beta_\eta^{(r-1)} + \partial_{\boldsymbol{\eta}_{(5-r)}}\beta_\eta \mathbf{v}_\eta. \quad (14)$$

By Corollary 4.3, $\partial_{\boldsymbol{\eta}_{(5-r)}}\beta_\eta$ is full rank and we can design

$$\mathbf{v}_\eta = (\partial_{\boldsymbol{\eta}_{(5-r)}}\beta_\eta)^\perp \left(- \sum_{j=1}^{r-1} \partial_{\boldsymbol{\eta}_j}\beta_\eta^{(r-1)} - \mathbf{K}_\eta \tilde{\beta}_\eta \right) \quad (15)$$

that annihilates it, where $\tilde{\beta}_\eta := \text{col}(\beta_\eta, \dot{\beta}_\eta, \dots, \beta_\eta^{(r-1)})$, and \mathbf{K}_η is a positive-definite gain matrix of an appropriate size.

Together Proposition 4.1 and Proposition 4.2 provide sufficient conditions on the position-related constraint function β_η under which the controller (15) asymptotically stabilizes the set \mathcal{P}^* relative to Γ^* and makes it invariant. In other words, the objective **PC** is satisfied.

B. $\boldsymbol{\mu}$ -coordination

In this section, we discuss the coordination among $\boldsymbol{\mu}$ states of each agent, which corresponds to their heading direction and its derivative. To this end, we consider the coordination map $\beta_\mu : \Psi_\mu(\mathcal{O}) \subseteq \mathbb{R}^{\dim(\mathcal{P}^*)} \rightarrow \mathbb{R}^d$ given in Definition 2.2 and the corresponding $\boldsymbol{\mu}$ -coordination set

$$\mathcal{H} := \{ \boldsymbol{\mu} \in \mathcal{P}^* : \beta_\mu(\boldsymbol{\mu}) = 0 \}.$$

We represent the above set with the letter \mathcal{H} as we use this set to synchronize heading and its derivative for the multi-agent system. To satisfy the general coordination condition $\beta_\mu(\boldsymbol{\mu}) = 0$, there may or may not exist a suitable coordination controller, and at times this leads to practically non-feasible coordination scenarios. Therefore, we restrict the subsequent discussion to the following special cases of the map β_μ .

Proposition 4.4: For $\boldsymbol{\mu} \in \mathcal{H}$, there exists a nonempty controlled-invariant set \mathcal{H}^* nested in \mathcal{H} if either of the following conditions hold:

- 1) **Heading coordination:** $\partial_{\boldsymbol{\mu}_2}\beta_\mu = 0$, and

$$\text{Im}(\partial_{\boldsymbol{\mu}_1}\beta_\mu) \ni \partial_{\boldsymbol{\mu}_1}(\partial_{\boldsymbol{\mu}_1}\beta_\mu)\boldsymbol{\mu}_2. \quad (16)$$

- 2) **Heading rate coordination:** $\partial_{\boldsymbol{\mu}_1}\beta_\mu = 0$.

Moreover, for the first case, largest controlled-invariant set $\mathcal{H}^* = \{ \boldsymbol{\mu} \in \mathcal{H} : \dot{\beta}_\mu = 0 \}$, and for the second case $\mathcal{H}^* = \mathcal{H}$.

Corollary 4.5: For the coordination function $\beta_\mu : \Psi_\mu(\mathcal{O}) \subseteq \mathbb{R}^{\dim(\mathcal{P}^*)} \rightarrow \mathbb{R}^d$ and a fixed $i \in \{1, 2\}$, assuming that $\text{rank} \partial_{\boldsymbol{\mu}_i}\beta_\mu = d$, and $\partial_{\boldsymbol{\mu}_k}\beta_\mu = 0$, for $k \in \{1, 2\} \setminus i$, the vector relative degree of the $\boldsymbol{\mu}$ -subsystem (11) with respect to the function β_μ is given by $(3-i)\mathbf{1}_N$.

Given a heading-related constraint function β_μ , with corresponding vector-relative degree $r\mathbf{1}_N$, Corollary 4.5 allows us to express its r^{th} derivative as follows:

$$\beta_\mu^{(r)} = \sum_{j=1}^{r-1} \partial_{\boldsymbol{\mu}_j}\beta_\mu^{(r-1)} + \partial_{\boldsymbol{\mu}_{(3-r)}}\beta_\mu \mathbf{v}_\mu. \quad (17)$$

By Corollary 4.5, $\partial_{\boldsymbol{\mu}_{(3-r)}}\beta_\mu$ is full rank and we can design \mathbf{v}_μ that annihilates it

$$\mathbf{v}_\mu = (\partial_{\boldsymbol{\mu}_{(3-r)}}\beta_\mu)^\perp \left(- \sum_{j=1}^{r-1} \partial_{\boldsymbol{\mu}_j}\beta_\mu^{(r-1)} - \mathbf{K}_\mu \tilde{\beta}_\mu \right), \quad (18)$$

where $\tilde{\beta}_\mu := \text{col}(\beta_\mu, \dot{\beta}_\mu, \dots, \beta_\mu^{(r-1)})$, and \mathbf{K}_μ is a positive-definite gain matrix of an appropriate size.

Proposition 4.4 provides sufficient conditions on the heading-related constraint function β_μ under which the controller (18) asymptotically stabilizes \mathcal{H}^* relative to \mathcal{P}^* and makes it invariant. In other words, the objective **HC** is satisfied. Finally, since we have established that Γ^* is locally asymptotically stable and \mathcal{P}^* is locally asymptotically stable relative to Γ^* , and both Γ^* and \mathcal{P}^* are compact, Theorem 1.1 implies that \mathcal{P}^* is locally asymptotically stable. Similarly, since both \mathcal{P}^* and \mathcal{H}^* are compact and positively invariant, and \mathcal{H}^* is locally asymptotically stable relative to \mathcal{P}^* , using Theorem 1.1 we can argue that \mathcal{H}^* is locally asymptotically stable. In other words, the final objective **AS** is also satisfied, and hence the multi-agent path following problem is solved.

V. SIMULATION RESULTS

In this mission three agents are required to follow three concentric circles at the height of ten meters. In Figure 1, the initial position of each agent is indicated by a solid coloured dot, and the agents converge to the corresponding coloured circle and follows it. In other words, the multi-agent system converge to the multi-agent path following manifold Γ^* , and stays there for all future time. Hence **PF** is satisfied. Next, we define a position-related coordination function to synchronize velocity of each agent, i.e., $\beta_\eta = \text{col}(\eta_2^1 + 0.3, \eta_2^2 - \eta_2^1, \eta_2^3 - \eta_2^2)$. We invoke the coordination controller (15) at $t = 15$ sec. As seen in Figure 2, initially the agents were following the paths in a non-coordinated manner. However, at $t = 15$ sec, each agent starts converging to the desired velocity constraint β_η while each agent stays on the path. In other words, the multi-agent system converges to the set $\mathcal{P}^* \subset \Gamma^*$ and hence **PC** is satisfied. Finally, we demonstrate the heading-related

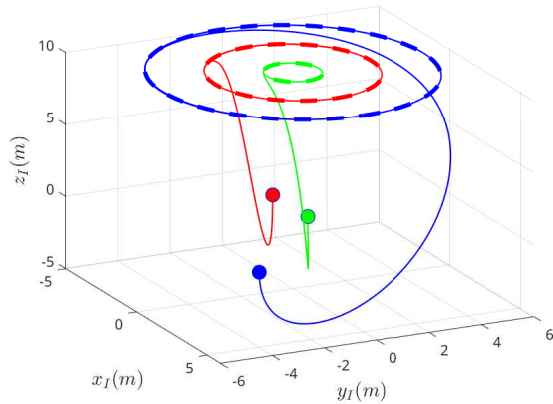


Fig. 1. Three UAVs performing a coordinated path following task

coordination and to achieve this we define the heading related constraint $\beta_\mu = \text{col}(\mu_1^1 - 0, \mu_1^2 - \mu_1^1 + \pi/4, \mu_1^3 - \mu_1^2 - \pi/4)$. We invoke the heading-related synchronized controller (18) at $t = 25$ secs. Because of space limitations, we are unable to show all the plot here; however, a complete video of the mission along with detailed simulations plots are available

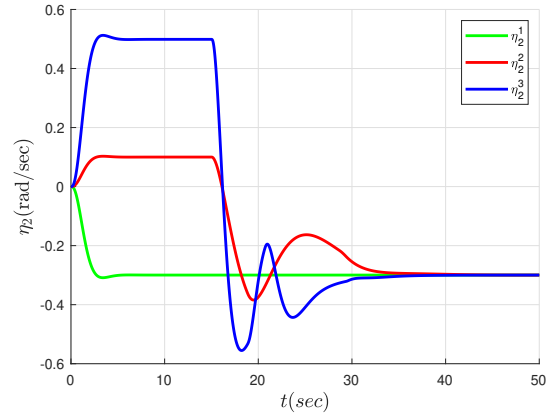


Fig. 2. Three UAVs demonstrate velocity coordination when the coordination controller is activated at $t = 15$ sec

here³. In summary, the multi-agent system converges to the invariant set \mathcal{H}^* which implies that **HC** is also satisfied. Finally, reduction theorem guarantees the asymptotic stability of the overall multi-agent system, i.e., **AS** is also satisfied.

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³<https://adeelakhtar.com/projects/MAS/>