

Controller Class for Rigid Body Tracking on $SO(3)$

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Abstract—In this paper, we propose a novel family of Lie algebra valued functions \mathcal{F}_R on special orthogonal group $SO(3)$. Each function belonging to this family induces a novel controller that stabilizes a rigid body attitude. All controllers induced by the family \mathcal{F}_R form a controller class \mathcal{C}_R . This novel controller class \mathcal{C}_R contains both local and almost-global asymptotically stable controllers, and this paper presents geometric stability results of the whole controller class \mathcal{C}_R .

I. INTRODUCTION

Rigid body attitude control is the key part of most of the control problems in the field of aerospace and robotics systems, such as aerial robots, satellites, and underwater vehicles. An orientation of a rigid body, also called an attitude of a rigid body, belongs to the set of 3×3 orthogonal matrices. This set forms a group, a Lie group $SO(3)$, which is also a smooth manifold. Informally, a rigid body control problem requires achieving the desired orientation, starting from any arbitrary orientation. This problem is generally referred to as an attitude tracking problem. It should be noted that, when the desired attitude is constant (i.e., does not change with respect to time), the attitude tracking problem reduces to attitude stabilization problem. Since the configuration space is not a Euclidean space \mathbb{R}^n , but a smooth manifold, standard control techniques designed for systems whose state space is in \mathbb{R}^n are not directly applicable. A natural way to overcome this difficulty is to find a local parameterization, or a local chart, of $SO(3)$. However, local parameterization, as the name implies, leads to local results and suffers other limitations such as gimbal lock [1]–[4]. Global results can be achieved by selecting multiple charts such that their union covers the whole $SO(3)$. Still, the major limitation is that the resulting controllers are not smooth and must perform switching [5].

Nonlinear geometric control techniques can overcome the shortcomings of the attitude control methodologies mentioned above, i.e., using a local parameterization or multiple charts. In [3], the authors consider the problem of attitude stabilization in great detail and underscore the fact that no continuous time-invariant feedback controller can globally asymptotically stabilize the desired attitude equilibrium. The authors also present controllers for a rigid body represented in terms of both full attitude and reduced attitude. A reduced attitude control problem deals with the configuration of a rigid body defined on a two-sphere \mathbb{S}^2 . It is possible to design a global feedback controller for attitude stabilization using hybrid control techniques [6], but the limitation lies in the non-smooth nature

of the controllers. In our work, we focus on the full attitude geometric control problem and build a framework to design smooth controllers.

The geometric attitude control problem can also be divided into a local geometric control problem or an almost-global geometric control problem. Here, we want to distinguish between a local controller that is designed using a local-chart and a local geometric controller that is coordinate-free. Although both controllers are local in terms of region of convergence, the latter is coordinate-free, and offer certain advantages such as avoiding gimbal lock. In contrast, almost-global geometric controllers are not only coordinate-free but also offer a region of convergence that constitute the whole state space except a “small” set of zero measure. In this work, we propose a rich class of geometric, or coordinate-free, attitude controllers that consist of local geometric controllers and almost-global geometric controllers. Depending on the application, a suitable controller can be selected.

One of the key ingredients for the geometric attitude control design for systems defined on $SO(3)$ is the notion of error function on $SO(3)$, which, in [7], is defined as a real-valued map. In [8], F. Bullo and R. Murray extend the definitions of these error functions given by [7], and propose real-valued left- and right-position error functions, and real-valued left- and right-velocity error functions. Moreover, a general framework for the control of the fully-actuated Lagrangian system is presented in [8]. The real-valued error functions proposed by F. Bullo and R. Murray serve as the foundation of a major portion of research in the area of geometric attitude and tracking control on $SO(3)$, as the researchers used the same error functions [9]–[11] or slightly modified version of real-valued functions [6], [12], [13]. Moreover, when solving the geometric control problem on the special Euclidean group of under-actuated systems, the researchers rely on the same real-valued error function [8] proposed by Bullo et al. to solve the attitude (inner) loop control of the underactuated system [4], [14], [15]. Unlike the previously mentioned approaches, which are based on a real-valued error function, we propose a novel family of Lie algebra valued functions. This family of functions induces a rich geometric controller class that stabilizes the fully-actuated rigid body system. In [16], [17], the authors use Lie algebra valued maps, but they consider a first-order system, while in this article, we deal with a second-order system. Moreover, our proposed family is different, and to the best of our knowledge is novel. Following is the list of our contributions:

- 1) a novel family of Lie algebra valued functions;
- 2) a novel geometric controller class that consists of several almost-global geometric controllers, and several local geometric controllers;

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- 3) a proof that shows the stability of each controller belonging to this class of controllers.

A. Notation and Math Preliminaries

Given a manifold M and a point $p \in M$, the tangent space of M at this point p is denoted by $T_p M$. A curve, or trajectory, passing through p is a map $\gamma : I \rightarrow M$, where I is the interval containing 0 for which $\gamma(0) = p$. The symbol $:=$ is used to represent equal by definition. An element $x \in \mathbb{R}^n$ is, by definition, n -tuple (x_1, \dots, x_n) and its Euclidean norm is represented by $\|x\|$. For a matrix $A \in \mathbb{R}^{n \times n}$, the 2-norm and Frobenius norm are represented by $\|A\|_2$ and $\|A\|_F$, respectively. Let 0_n represent a column vector of dimension n with all zero elements, and I_n (or alternatively I) represent an $n \times n$ identity matrix. The trace of a matrix $A \in \mathbb{R}^{n \times n}$ is represented by $\text{trace}(A)$ and its transpose is denoted by $A^\top \in \mathbb{R}^{n \times n}$. For any natural number $n \in \mathbb{N}$, a unit n -sphere is given by $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$.

Consider a rigid body capable of rotating, without loss of generality, about its center of gravity. Let \mathcal{I} denote the fixed reference frame, and let \mathcal{B} be a frame attached to its center of mass. Then, the orientation of the rigid body belongs to the set

$$\text{SO}(3) = \{R \in \mathbb{R}^{3 \times 3} : R^\top R = I, \det(R) = 1\},$$

which has a Lie group structure [18]. Its associated Lie algebra is the real vector space of 3×3 skew symmetric matrices

$$\mathfrak{so}(3) = \{A \in \mathbb{R}^{3 \times 3} : A = -A^\top\}$$

equipped with the matrix commutator $[A, B] := AB - BA$ as its binary operation. The $\mathfrak{so}(3)$, with a vector space structure, is isomorphic to \mathbb{R}^3 . The isomorphism [18] is denoted by $\hat{\cdot} : \mathbb{R}^3 \rightarrow \mathfrak{so}(3)$ and its inverse is denoted by $(\cdot)^\vee : \mathfrak{so}(3) \rightarrow \mathbb{R}^3$ such that, for any $v, w \in \mathbb{R}^3$, $\hat{v}w = v \times w$, where \times is the standard vector (cross) product in \mathbb{R}^3 . Body and spatial angular velocities express the same physical quantity represented in different reference frames. Given a rigid body with orientation $R \in \text{SO}(3)$, the adjoint map provides a transformation from a body reference frame to the spatial reference and is given by

$$\begin{aligned} \text{Ad}_R : \mathfrak{so}(3) &\rightarrow \mathfrak{so}(3) \\ A &\mapsto RAR^\top. \end{aligned} \quad (1)$$

Finally, the matrix exponential is an analytic diffeomorphism [18] between $U_{\mathfrak{so}(3)} := \{\hat{\omega} \in \mathfrak{so}(3) : \omega \in \mathbb{R}^3, \|\omega\|_2 < \pi\}$ and $U_{\text{SO}(3)} := \{R \in \text{SO}(3) : \text{trace}(R) \neq -1\}$. The inverse map from $U_{\text{SO}(3)} \rightarrow U_{\mathfrak{so}(3)}$ is the principal matrix logarithm and is denoted by Log . For $x \in \mathbb{R}^3$ and $A \in \mathbb{R}^{3 \times 3}$, the following property holds [14]:

$$\hat{x}A + A^\top \hat{x} = (\{\text{trace}(A)I - A\}x)^\wedge. \quad (2)$$

¹For $v \in \mathbb{R}^3$, the corresponding element of Lie algebra is given by \hat{v} , or, whenever it is syntactically convenient, by $(v)^\wedge$.

II. MATHEMATICAL MODEL

We consider a fully-actuated rigid body capable of rotating about its center of gravity without translation in the three-dimensional space. It is assumed that the system is subject to three body-fixed control inputs aligned with the three principal axes. The kinematic equation governing the trajectory $t \mapsto R(t)$ is given by

$$\dot{R}(t) = R(t)\hat{\Omega}(t), \quad (3)$$

where $t \mapsto \Omega(t) \in \mathbb{R}^3$ is the body angular velocity² [18], [19]. The Euler-Poincaré equations describing the trajectory $t \mapsto \Omega(t)$ are

$$J\dot{\Omega}(t) = \tau(t) - (\Omega(t) \times J\Omega(t)), \quad (4)$$

where $J \in \mathbb{R}^{3 \times 3}$ is the inertia of the fully-actuated system in the body-fixed frame and $t \mapsto \tau(t) := (\tau_p(t), \tau_q(t), \tau_r(t))$ denote the total moments acting on the rigid body. Together equations (3) and (4) constitute the dynamic model of the rigid body. We apply a preliminary feedback to the system

$$\tau(t) = \Omega(t) \times J\Omega(t) + Ju(t),$$

where $t \mapsto u(t) \in \mathbb{R}^3$ is an auxiliary control input to be designed. This feedback simplifies (4) and we obtain

$$\dot{\Omega}(t) = u(t). \quad (5)$$

Given the attitude dynamics of a rigid body as in (3) and (5), and a desired attitude $R_d : [0, \infty) \rightarrow \text{SO}(3)$ generated by the exogenous system

$$\begin{aligned} \dot{R}_d(t) &= R_d(t)\hat{\Omega}_d(t), \\ \dot{\Omega}_d(t) &= u_d(t), \end{aligned} \quad (6)$$

where $u_d : [0, \infty) \rightarrow \mathbb{R}^3$ is assumed to be continuously differentiable, we seek a feedback control law for the input u in (5) such that $R(t)$ asymptotically approaches $R_d(t)$ when the initial tracking error $R(0) - R_d(0)$ is sufficiently small. We further assume that the reference frame $R_d(t)$ does not move “too fast” in a sense to be made precise later. We allow our controllers to depend on the plant states R, Ω as well as the exogenous states R_d, Ω_d and u_d . Thus we call this an attitude tracking problem with full information.

The distance on $\text{SO}(3)$ cannot be defined in a usual Euclidean sense, because $\text{SO}(3)$ is not closed under addition. To compare the “difference” between two rotation matrices $R_1, R_2 \in \text{SO}(3)$, a distance function or a metric on $\text{SO}(3)$ can be defined as [20]

$$\begin{aligned} \Lambda : \text{SO}(3) \times \text{SO}(3) &\rightarrow [0, 2\sqrt{2}] \subset \mathbb{R} \\ (R_1, R_2) &\mapsto \|I - R_1^\top R_2\|_F. \end{aligned} \quad (7)$$

It should be noted that Λ gives a measure of rotation that is required to be applied to R_1 to align it with R_2 .

²The body angular velocity can be thought of as a map [17], given by $\hat{\Omega} : \text{SO}(3) \rightarrow \mathfrak{so}(3)$, $R \mapsto R^\top \dot{R}$.

A. Problem statement

Consider the rigid body represented by (3) and (5), and an exogenous system given by (6), we seek a smooth feedback control law

$$u : (\text{SO}(3))^2 \times (\mathfrak{so}(3))^2 \rightarrow \mathbb{R}^3,$$

such that if $\|I - R_d^\top R\|_F \leq d_\Lambda$, for some $d_\Lambda \in [0, 2\sqrt{2}]$ and for all $t \geq 0$, the rigid body satisfies the following tracking goals:

- T1** the attitude of the rigid body tracks the desired attitude of the exogenous system, i.e., $\|I - R_d^\top R\|_F \rightarrow 0$, as $t \rightarrow \infty$;
- T2** the body rates of the rigid body tracks the desired body rates of the exogenous system, i.e., $\|\Omega - \Omega_d\| \rightarrow 0$, as $t \rightarrow \infty$.

This problem is referred to as the attitude tracking problem. We characterize d_Λ precisely in the sections to follow.

III. CLASS OF FEEDBACK CONTROLLERS

To solve the attitude tracking problem, we design a class of feedback controllers. First, we present a novel notion of Lie algebra valued functions.

A. Function family \mathcal{F}_R

We propose a family of functions \mathcal{F}_R that induces a controller class. We call this family \mathcal{F}_R because it consists of functions that depend only on position, i.e., rotation matrices R and R_d . We define a domain set $\mathbb{D}_{\text{SO}(3)}$ consisting of all the points in the product space of $\text{SO}(3) \times \text{SO}(3)$ such that the distance between two points does not achieve maximum value on the metric Λ

$$\mathbb{D}_{\text{SO}(3)} := \left\{ (R, R_d) \in (\text{SO}(3))^2 : \|I - R_d^\top R\|_F \neq 2\sqrt{2} \right\}.$$

An alternative way to define the same set is

$$\mathbb{D}_{\text{SO}(3)} := \left\{ (R, R_d) \in (\text{SO}(3))^2 : \text{trace}(R_d^\top R) \neq -1 \right\}.$$

Definition III.1. A function $f : U \subseteq \mathbb{D}_{\text{SO}(3)} \rightarrow \mathfrak{so}(3)$ is said to belong to \mathcal{F}_R if there exists an open set $U \subseteq \mathbb{D}_{\text{SO}(3)}$ containing (I, I) such that

- P1** it is twice continuously differentiable on U ,
- P2** $f^{-1}(0) = \{(R, R_d) \in U : R = R_d\}$, and
- P3** the differential of $\frac{d}{dt}f(R(t), R_d(t))$ with respect to $\widehat{\Omega}(t)$ is nonsingular on $U \times (\mathfrak{so}(3))^2$ at least in the neighborhood of the desired point, which is $R(t) = R_d(t)$ and $\Omega(t) = \Omega_d(t)$.

Some examples of functions belonging to the \mathcal{F}_R family are

- $f = (R_d(t)^\top R(t) - R(t)^\top R_d(t))$,
- $f = \text{Log}(R_d(t)^\top R(t))$,
- $f = \text{Log}(R(t)) - \text{Log}(R_d(t))$, and
- $f = \text{trace}(I - R(t)^\top R_d(t))(R_d(t)^\top R(t) - R(t)^\top R_d(t))$.

The first two examples are discussed in detail in Section IV.

B. \mathcal{C}_R controller class

Let f be any function in \mathcal{F}_R . With a slight abuse of notation, we omit the argument t of R, R_d, Ω , and Ω_d for simplification of notation. We start by taking the Lie derivative of $f \in \mathcal{F}_R$ along the vector fields of (3), (5), and (6). This yields

$$\begin{aligned} \frac{d}{dt}f(R, R_d) &= (d_R f)\dot{R} + (d_{R_d} f)\dot{R}_d, \\ &= (d_R f)R\widehat{\Omega} + (d_{R_d} f)R_d\widehat{\Omega}_d. \end{aligned} \quad (8)$$

Since the control input does not appear, we take the second derivative of f . Direct calculations give the following result

$$\begin{aligned} \frac{d^2}{dt^2}f(R, R_d) &= (d_R(d_R f))R\widehat{\Omega}R\widehat{\Omega} + (d_R f)R\widehat{\Omega}\widehat{\Omega} \\ &\quad + (d_R(d_{R_d} f))R\widehat{\Omega}R_d\widehat{\Omega}_d + (d_{R_d}f)R_d\widehat{\Omega}_d\widehat{\Omega}_d \\ &\quad + (d_{R_d}(d_R f))R_d\widehat{\Omega}_dR\widehat{\Omega} \\ &\quad + (d_{R_d}(d_{R_d} f))R_d\widehat{\Omega}_dR_d\widehat{\Omega}_d \\ &\quad + (d_R f)R\widehat{u} + (d_{R_d} f)R_d\widehat{u}_d. \end{aligned}$$

By property **P3**, $d_R f$ is nonsingular when $R = I$. Therefore, by property **P1**, it is invertible in a neighbourhood of $R = I$. Thus the feedback controller

$$\begin{aligned} \widehat{u} &:= ((d_R f)R)^{-1} \left\{ - (d_{R_d} f)R_d\widehat{u}_d \right. \\ &\quad - (K_1 f^\vee)^\wedge - (K_2 \dot{f}^\vee)^\wedge \\ &\quad - (d_R(d_R f))R\widehat{\Omega}R\widehat{\Omega} - (d_R f)R\widehat{\Omega}\widehat{\Omega} \\ &\quad - (d_R(d_{R_d} f))R\widehat{\Omega}R_d\widehat{\Omega}_d - (d_{R_d}(d_R f))R_d\widehat{\Omega}_dR\widehat{\Omega} \\ &\quad \left. - (d_{R_d}(d_{R_d} f))R_d\widehat{\Omega}_dR_d\widehat{\Omega}_d - (d_{R_d}f)R_d\widehat{\Omega}_d\widehat{\Omega}_d \right\}, \end{aligned} \quad (9)$$

where $K_1, K_2 \in \mathbb{R}^{3 \times 3}$ are Hurwitz, is well-defined in a neighbourhood of $R = I$. We call the control law (9) a class \mathcal{C}_R feedback controller. We want to underscore that the controller class \mathcal{C}_R is designed without any local chart. In other words, the controller is designed directly on the manifold $\text{SO}(3)$, therefore the controller class \mathcal{C}_R is geometric. Now we prove the main result.

Theorem III.2. Given rigid body dynamics in (3) and (5), and an exogenous system satisfying (6), each controller in the class \mathcal{C}_R locally asymptotically stabilizes the rigid body on $\mathbb{D}_{\text{SO}(3)} \times (\mathfrak{so}(3))^2$.

Proof. Let $f(R, R_d)$ belong to the family \mathcal{F}_R . The time derivative of $f(R, R_d)$ is given by

$$\frac{d}{dt}f(R, R_d) := g = (d_R f)R\widehat{\Omega} + (d_{R_d} f)R_d\widehat{\Omega}_d. \quad (10)$$

We pick a real-valued positive definite Lyapunov function

$$\begin{aligned} V : ((\text{SO}(3))^2 \times (\mathfrak{so}(3))^2) &\rightarrow \mathbb{R} \\ (R, R_d, \widehat{\Omega}, \widehat{\Omega}_d) &\mapsto \frac{1}{2}\|g^\vee\|_2^2 + \frac{1}{2}\|K_1 f^\vee\|_2^2. \end{aligned} \quad (11)$$

We take the derivative of the Lyapunov function

$$\begin{aligned}\dot{V} &= (g^\vee)^\top \frac{d}{dt} g^\vee + (K_1 f^\vee)^\top \frac{d}{dt} f^\vee \\ &= (g^\vee)^\top \left\{ (d_R(d_R f)) R \widehat{\Omega} R \widehat{\Omega} + (d_R f) R \widehat{\Omega} \widehat{\Omega} \right. \\ &\quad + (d_R(d_{R_d} f)) R \widehat{\Omega} R_d \widehat{\Omega}_d + (d_{R_d}(d_R f)) R_d \widehat{\Omega}_d R \widehat{\Omega} \\ &\quad + (d_{R_d}(d_{R_d} f)) R_d \widehat{\Omega}_d R_d \widehat{\Omega}_d + (d_{R_d f}) R_d \widehat{\Omega}_d \widehat{\Omega}_d \\ &\quad \left. + (d_R f) R \widehat{u} + (d_{R_d} f) R_d \widehat{u}_d \right\}^\vee + (K_1 f^\vee)^\top (g^\vee).\end{aligned}$$

By applying the controller \mathcal{C}_R from (9) in the above expression, we get

$$\begin{aligned}\dot{V} &= (g^\vee)^\top \left\{ - (K_1 f^\vee)^\wedge - (K_2 \dot{f}^\vee)^\wedge \right\}^\vee + (K_1 f^\vee)^\top (g^\vee) \\ &= (g^\vee)^\top \left\{ - (K_1 f^\vee) - (K_2 \dot{f}^\vee) \right\} + (K_1 f^\vee)^\top (g^\vee) \\ &= (g^\vee)^\top \left\{ - (K_1 f^\vee) - (K_2 g^\vee) \right\} + (K_1 f^\vee)^\top (g^\vee) \\ &= - (g^\vee)^\top (K_1 f^\vee) - (g^\vee)^\top (K_2 g^\vee) + (K_1 f^\vee)^\top (g^\vee),\end{aligned}\tag{12}$$

which is simplified to

$$\begin{aligned}\dot{V} &= - (K_1 f^\vee)^\top (g^\vee) - (g^\vee)^\top (K_2 g^\vee) + (K_1 f^\vee)^\top (g^\vee) \\ &= - \|\sqrt{K_2} g^\vee\|_2^2.\end{aligned}\tag{13}$$

Hence, by [21, Theorem 4.1], the system is locally asymptotically stable on $\mathbb{D}_{\text{SO}(3)} \times (\mathfrak{so}(3))^2$. Moreover, the controller is at most almost-globally asymptotically stable on $\mathbb{D}_{\text{SO}(3)} \times (\mathfrak{so}(3))^2$. Next, we give an invariance argument to prove stability. Let

$$\Gamma := \{(R, R_d) \in U \subseteq \mathbb{D}_{\text{SO}(3)} : \|f(R, R_d)\| \leq c\},$$

for some positive constant c . By Definition III.1, one can always pick a function such that f is bounded by some constant c . The following set is invariant by the construction of the controller design; moreover, it is the largest invariant set

$$\Gamma^* := \{(R, R_d) \in U \subseteq \mathbb{D}_{\text{SO}(3)} : f = \dot{f} = \ddot{f} = \widehat{0}_3\}.$$

Equivalently, we can write

$$\overline{\Gamma}^* := \{(R, R_d) \in U \subseteq \mathbb{D}_{\text{SO}(3)} : \|f^\vee\| = \|\dot{f}^\vee\| = \|\ddot{f}^\vee\| = 0\}.$$

It is easy to see that $\overline{\Gamma}^* \subset \Gamma$, and by P2, $\overline{\Gamma}^* = \{(R, R_d) \in U \subseteq \mathbb{D}_{\text{SO}(3)} : R = R_d\}$. Since $\overline{\Gamma}^*$ is the largest control invariant set contained in Γ , therefore, whenever $(R(0), R_d(t)) \in \Gamma$, for all t , all system states converge to $\overline{\Gamma}^*$ as $t \rightarrow \infty$. \square

Under the light of Theorem III.2, each controller in the \mathcal{C}_R controller class is stable, and satisfies T1 and T2. Hence, the attitude tracking problem is solved.

IV. APPLICATIONS OF \mathcal{C}_R CONTROLLERS

In this section, we present two detailed examples of the \mathcal{C}_R controllers induced by the \mathcal{F}_R family of functions.

Example IV.1. Consider the function

$$\begin{aligned}f : U \subseteq \mathbb{D}_{\text{SO}(3)} &\rightarrow \mathfrak{so}(3) \\ (R(t), R_d(t)) &\mapsto \text{Log}(R_d(t)^\top R(t)).\end{aligned}$$

Take the open set $U = \mathbb{D}_{\text{SO}(3)}$. Since the matrix logarithm is analytic in U , P1 holds. Also, it is clear that $f^{-1}(0)$ is the subset of U in which $R(t) = R_d(t)$. To check P3, we take the derivative³ of f

$$\begin{aligned}\frac{d}{dt} f(R(t), R_d(t)) &= \widehat{\Omega}(t) \\ &\quad - (R(t)^\top R_d(t)) \widehat{\Omega}_d(t) (R_d(t)^\top R).\end{aligned}\tag{14}$$

From the expression above, it is easy to check that $d_{\widehat{\Omega}}(\frac{d}{dt} f(R(t), R_d(t)))$ is nonsingular everywhere in U , satisfying P3. Before proceeding further, we drop the argument t and the arguments of $f(R, R_d)$ for the convenience of notation. Following the controller design procedure, we take the second derivative of (14)

$$\begin{aligned}\frac{d^2}{dt^2} f &= \widehat{\Omega} - \left\{ \frac{d}{dt} (R^\top R_d) \right\} \widehat{\Omega}_d (R_d^\top R) \\ &\quad - (R^\top R_d) \widehat{\Omega}_d (R_d^\top R) \\ &\quad - (R^\top R_d) \widehat{\Omega}_d \left\{ \frac{d}{dt} (R_d^\top R) \right\}.\end{aligned}\tag{15}$$

Using the expressions given in Appendix VII, the above equation can be written as

$$\begin{aligned}\frac{d^2}{dt^2} f &= \widehat{u} - (R^\top R_d) \widehat{\Omega}_d (R_d^\top R) \\ &\quad - \left\{ (R^\top R_d) \widehat{\Omega}_d - \widehat{\Omega} (R^\top R_d) \right\} \widehat{\Omega}_d (R_d^\top R) \\ &\quad - (R^\top R_d) \widehat{\Omega}_d \left\{ (R_d^\top R) \widehat{\Omega} - \widehat{\Omega}_d (R_d^\top R) \right\}.\end{aligned}\tag{16}$$

It can be seen that the term multiplied by \widehat{u} is I , therefore by property P1 it is invertible everywhere. The closed form expression of the controller can be written as

$$\begin{aligned}\widehat{u} &= (R^\top R_d) \widehat{\Omega}_d (R_d^\top R) \\ &\quad + \left\{ (R^\top R_d) \widehat{\Omega}_d - \widehat{\Omega} (R^\top R_d) \right\} \widehat{\Omega}_d (R_d^\top R) \\ &\quad + (R^\top R_d) \widehat{\Omega}_d \left\{ (R_d^\top R) \widehat{\Omega} - \widehat{\Omega}_d (R_d^\top R) \right\} \\ &\quad - (K_1 f^\vee)^\wedge - (K_2 \dot{f}^\vee)^\wedge.\end{aligned}\tag{17}$$

It is easy to see that the controller $\widehat{u} \in \mathcal{C}_R$ is defined everywhere on $\mathbb{D}_{\text{SO}(3)}$. However, the controller is not defined everywhere on $(\text{SO}(3))^2 \times (\mathfrak{so}(3))^2$ because $f = \text{Log}(R_d^\top R)$ is not defined globally. Precisely, $f = \text{Log}(R_d^\top R)$ is not defined when $\text{trace}(R_d^\top R) = -1$. On $\text{SO}(3)$, $\text{trace}(R_d^\top R) = -1$ at the following three points:

- $(R_d^\top R) = \text{diag}(-1, -1, 1)$;
- $(R_d^\top R) = \text{diag}(-1, 1, -1)$;
- $(R_d^\top R) = \text{diag}(1, -1, -1)$.

Since these points form locally a set of common zeros of finitely many analytic functions, its measure is zero. By Theorem III.2, the controller is asymptotically stable everywhere except on the set of measure zero; hence, the controller is almost-globally asymptotically stable. \triangle

Remark IV.2. The controller \widehat{u} given in (17) is not globally stable. In fact, no continuous time-invariant feedback controller can globally asymptotically stabilize an equilibrium

³see Appendix VII for the closed form expressions of the derivatives.

attitude of a rigid body, or globally track a reference attitude because of topological obstructions [3], [22]. The strongest stability or tracking results that can be achieved is almost-global asymptotic stability. In terms of the region of convergence, the controller (17) is the strongest possible controller. It can be shown that other controllers belonging to the class \mathcal{C}_R also enjoy the almost-global asymptotic property.

Physically, the condition when $\text{trace}(R_d^\top R) = -1$ happens when the desired orientation is furthestmost apart from the current orientation. In other words, when the distance metric Λ achieves the maximum value $2\sqrt{2}$. More intuitively, $\text{trace}(R_d^\top R) = -1$, when, for example, a rigid body is upside down, and the desired orientation is upright. In local coordinates (Euler Angles), this condition happens when the desired orientation is π radians apart in either roll, pitch, or yaw axis.

Example IV.3. Consider another function from the \mathcal{F}_R family, i.e.,

$$f: U \subseteq \mathbb{D}_{\text{SO}(3)} \rightarrow \mathfrak{so}(3) \\ (R(t), R_d(t)) \mapsto R_d(t)^\top R(t) - R(t)^\top R_d(t),$$

with

$$U := \left\{ (R, R_d) \in \text{SO}(3)^2 : \Lambda < 2 \right\}.$$

First, we show that this function belongs to \mathcal{F}_R family. Similar to Example IV.1, one can check that this function satisfies both **P1** and **P2**. Using the results from Appendix VII, the first time derivative can be written as

$$\begin{aligned} \frac{d}{dt} f(R(t), R_d(t)) &= \frac{d}{dt} (R_d(t)^\top R(t) - R(t)^\top R_d(t)), \\ &= (R_d(t)^\top R(t)) \widehat{\Omega}(t) \\ &\quad - \widehat{\Omega}_d(t) (R_d(t)^\top R(t)) \\ &\quad + \widehat{\Omega}(R(t)^\top R_d(t)) \\ &\quad - (R(t)^\top R_d(t)) \widehat{\Omega}_d(t), \end{aligned} \quad (18)$$

which, by using property (2), can be written as

$$d_{\widehat{\Omega}} \left(\frac{d}{dt} f(R, R_d) \right) = \text{trace}(R^\top R_d) I - (R^\top R_d).$$

We evaluate $(\text{trace}(R^\top R_d) I - (R^\top R_d))$ at $(R = R_d)$

$$\begin{aligned} (\text{trace}(R^\top R_d) I - (R^\top R_d)) \Big|_{(R=R_d)} &= \text{trace}(I) I - I, \\ &= 3I - I, \\ &= 2I \in \text{GL}(3, \mathbb{R}). \end{aligned}$$

Since the above expression is invertible at $(R = R_d)$, by inverse function theorem [18] and continuity, $\{\text{trace}(R^\top R_d) I - (R^\top R_d)\}$ is invertible in the neighborhood of $(R = R_d)$. It can be seen that $\{\text{trace}(R^\top R_d) I - (R^\top R_d)\} \in \text{GL}(3, \mathbb{R})$ only when R is in a neighborhood of R_d . In other words, $d_{\widehat{\Omega}} \left(\frac{d}{dt} f(R, R_d) \right)$ is nonsingular in some neighborhood of $(R = R_d)$, hence **P3** is also satisfied. Next we show that this function induces a controller belonging to the controller

class \mathcal{C}_R . Following the controller design procedure, we take the second derivative of (18)

$$\begin{aligned} \frac{d^2}{dt^2} f(R, R_d) &= \left\{ \frac{d}{dt} (R_d^\top R) \right\} \widehat{\Omega} + (R_d^\top R) \widehat{\Omega} - \widehat{\Omega}_d (R_d^\top R) \\ &\quad - \widehat{\Omega}_d \left\{ \frac{d}{dt} (R_d^\top R) \right\} + \widehat{\Omega} \left\{ \frac{d}{dt} (R^\top R_d) \right\} \\ &\quad + \widehat{\Omega} (R^\top R_d) - \left\{ \frac{d}{dt} (R^\top R_d) \right\} \widehat{\Omega}_d \\ &\quad - (R^\top R_d) \widehat{\Omega}_d. \end{aligned}$$

Direct substitution of the expressions of the derivatives give

$$\begin{aligned} \frac{d^2}{dt^2} f(R, R_d) &= \left\{ (R_d^\top R) \widehat{\Omega} - \widehat{\Omega}_d (R_d^\top R) \right\} \widehat{\Omega} + (R_d^\top R) \widehat{\Omega} \\ &\quad - \widehat{\Omega}_d (R_d^\top R) - \widehat{\Omega}_d \left\{ (R_d^\top R) \widehat{\Omega} - \widehat{\Omega}_d (R_d^\top R) \right\} \\ &\quad + \widehat{\Omega} (R^\top R_d) + \widehat{\Omega} \left\{ (R^\top R_d) \widehat{\Omega}_d - \widehat{\Omega} (R^\top R_d) \right\} \\ &\quad - \left\{ (R^\top R_d) \widehat{\Omega}_d - \widehat{\Omega} (R^\top R_d) \right\} \widehat{\Omega}_d \\ &\quad - (R^\top R_d) \widehat{\Omega}_d. \end{aligned}$$

Combining the terms involving $\widehat{\Omega}$, and using (2), we can write

$$\begin{aligned} \frac{d^2}{dt^2} f(R, R_d) &= (\{\text{trace}(R^\top R_d) I - (R^\top R_d)\} u)^\wedge \\ &\quad - \widehat{\Omega}_d (R_d^\top R) - (R^\top R_d) \widehat{\Omega}_d \\ &\quad + \left\{ (R_d^\top R) \widehat{\Omega} - \widehat{\Omega}_d (R_d^\top R) \right\} \widehat{\Omega} \\ &\quad + \widehat{\Omega} \left\{ (R^\top R_d) \widehat{\Omega}_d - \widehat{\Omega} (R^\top R_d) \right\} \\ &\quad - \widehat{\Omega}_d \left\{ (R_d^\top R) \widehat{\Omega} - \widehat{\Omega}_d (R_d^\top R) \right\} \\ &\quad - \left\{ (R^\top R_d) \widehat{\Omega}_d - \widehat{\Omega} (R^\top R_d) \right\} \widehat{\Omega}_d. \end{aligned} \quad (19)$$

By **P3**, the matrix $(\{\text{trace}(R^\top R_d) I - (R^\top R_d)\})$ is invertible in some neighborhood of $R = I$. Let $D := (\{\text{trace}(R^\top R_d) I - (R^\top R_d)\})$. It can be shown that the matrix D is not invertible for all $R, R_d \in \text{SO}(3)$. Moreover, it loses rank whenever the following conditions hold:

- 1) $\text{trace}(R^\top R_d) = 1$;
- 2) $\text{trace}(R^\top R_d) = -1$.

The condition when D loses rank is shown in Figure 1. At the start of the simulation, R is aligned with R_d , i.e., $R = R_d$. In other words, $\text{trace}(R^\top R_d) = 3$, then the rigid body is rotated about an arbitrary axis to 2π . As seen in Figure 1, the matrix determinant $\det(D)$ goes to zero, when $\text{trace}(R^\top R_d) = 1$, or $\text{trace}(R^\top R_d) = -1$, and this happens when the rotation value is $\pm\pi/2$ or π . Another way to interpret this condition is that, on the distance metric Λ , D loses rank whenever $\Lambda = 2$ or $\Lambda = 2\sqrt{2}$. This motivates us to pick a neighborhood around R_d such that the distance Λ between each point in the neighborhood and R_d is less than 2, i.e.,

$$U := \left\{ (R, R_d) \in \text{SO}(3)^2 : \Lambda < 2 \right\}.$$

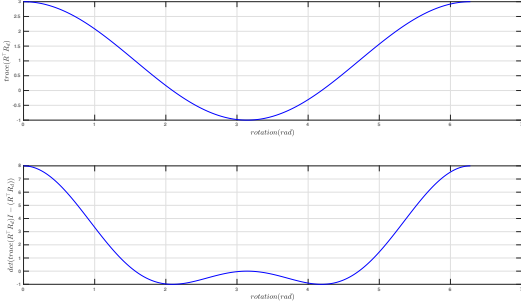


Fig. 1. The determinant $\det(D)$ goes to zero, when $\text{trace}(R^\top R_d) = 1$ or $\text{trace}(R^\top R_d) = -1$, and this happens when the rotation value is $\pm\pi/2$ or π .

Using **P1**, the closed form expression of the controller can be written as

$$\begin{aligned} \hat{u} = & \left(\left\{ \text{trace}(R^\top R_d)I - (R^\top R_d) \right\} \right)^{-1} \left[\hat{\Omega}_d (R_d^\top R) \right. \\ & - \left\{ (R_d^\top R) \hat{\Omega} - \hat{\Omega}_d (R_d^\top R) \right\} \hat{\Omega} \\ & - \hat{\Omega} \left\{ (R^\top R_d) \hat{\Omega}_d - \hat{\Omega} (R^\top R_d) \right\} \\ & + \hat{\Omega}_d \left\{ (R_d^\top R) \hat{\Omega} - \hat{\Omega}_d (R_d^\top R) \right\} + (R^\top R_d) \hat{\Omega}_d \\ & \left. + \left\{ (R^\top R_d) \hat{\Omega}_d - \hat{\Omega} (R^\top R_d) \right\} \hat{\Omega}_d (K_1 f^\vee)^\wedge - (K_2 \dot{f}^\vee)^\wedge \right]. \end{aligned} \quad (20)$$

By Theorem III.2, the controller (20) is asymptotically stable everywhere on U , and is local. \triangle

We need to highlight that although controller (20) is local, it is still geometric, i.e., the controller is designed in a coordinate-free manner.

V. SIMULATION

In this section, we present simulation results of the controller (17) presented in Example IV.1. Due to space limitations, the simulation results of the controller (20), designed in Example IV.3, are not presented in this article.

A. Stabilization

First, we discuss the case when R_d is time invariant. Without loss of generality, let $R_d = I$. Starting from an ‘‘almost’’ upside down position, i.e., the initial orientation at $t = 0$ is $R(0) = \exp((\pi - \epsilon)\hat{e}_1)$, where $|\epsilon| \approx 0 \in \mathbb{R}$ is close to zero but not identically zero. The target is to achieve upright position, i.e., $R = R_d = I$. It can be seen from Figure 2, that at $t = 0$ the error is almost $2\sqrt{2}$, which in other words is the almost upside down position. If the system starts from this almost-maximum error value, the error still converges to zero. The body errors $\Omega - \Omega_d$ about each body axis also converge to zero, as shown in Figure 3.

B. Sinusoidal signal tracking

Next, we consider a time varying reference signal R_d . Starting from an initial pose of $R(0) =$

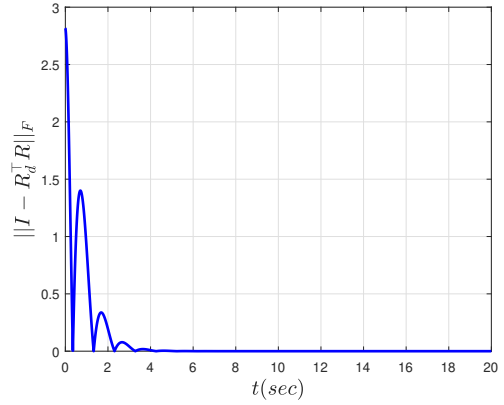


Fig. 2. Error between R and R_d

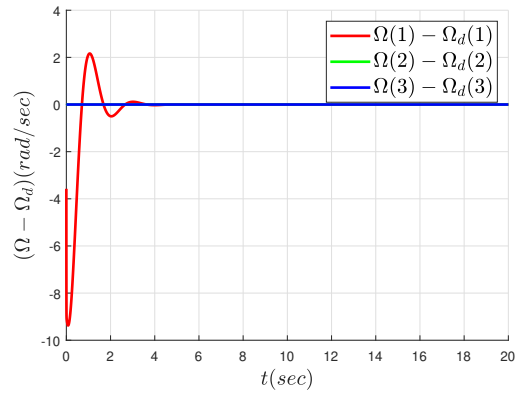


Fig. 3. Error between each body axis $\Omega - \Omega_d$

$\exp(((2\pi/3)e_1 + (2\pi/9)e_2 + (\pi/3)e_3)^\wedge)$, the target is to track the desired moving reference attitude

$$R_d(t) = \exp((170^\circ(\pi/180) \sin(0.001t))\hat{e}_1).$$

In local coordinates, the initial condition can be interpreted as an angle of $2\pi/3$, $2\pi/9$, and $\pi/3$ radians about roll, pitch and yaw axis, respectively, and the desired reference attitude R_d can be seen as a $\pm 170^\circ$ sinusoidal movement about the roll axis. It can be seen in Figure 4 that the tracking errors converge to zero. Figure 5 demonstrates the tracking performance of the rigid body in terms of local coordinates. We want to underscore that the results are shown in terms of Euler angles just for the purpose of demonstration, as it is relatively easy to visualize pose of a rigid body in terms of Euler angles. It can be seen that the rigid body crosses the so called gimbal lock point. Finally, Figure 6 shows that all body rate errors converge to zero.

C. Multiple flips with noise

Next, we demonstrate the performance of this controller in the presence of sensor noise. We consider noise levels of a low-cost sensor (inertial measurement unit), which is typically mounted on quadrotors, for details see [23]. Again, starting from an almost upside down position, the target is to track the reference $R_d(t) = \exp(0.005t\hat{e}_1)$ in the presence of noise.

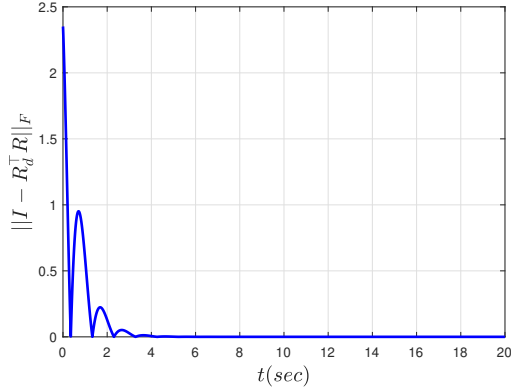


Fig. 4. Error between R and R_d

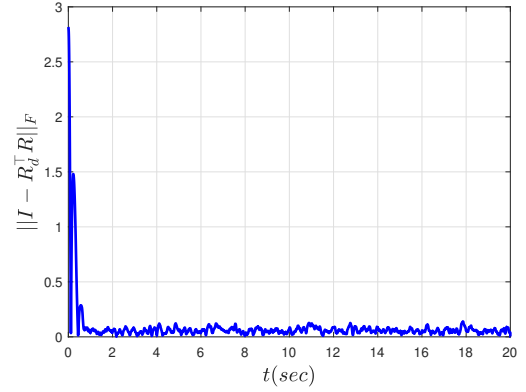


Fig. 7. Error between R and R_d

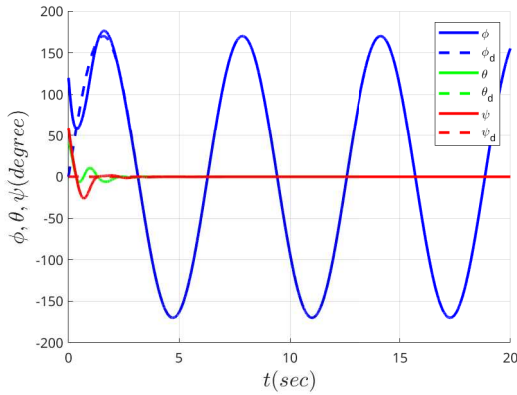


Fig. 5. Euler angles tracking the desired signals

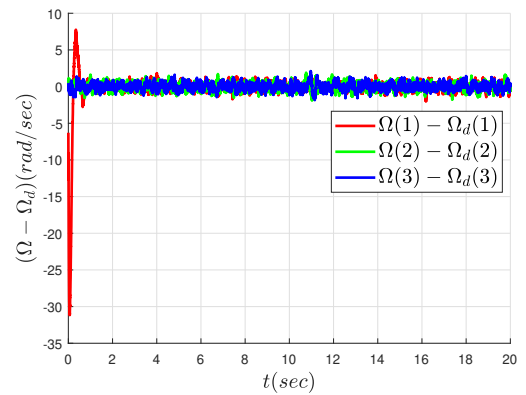


Fig. 8. Error between each body axis $\Omega - \Omega_d$ in the presence of noise

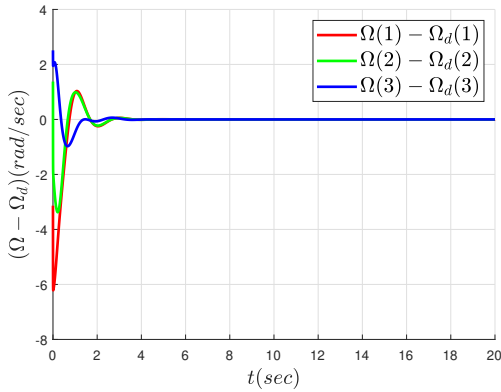


Fig. 6. Error between each body axis $\Omega - \Omega_d$

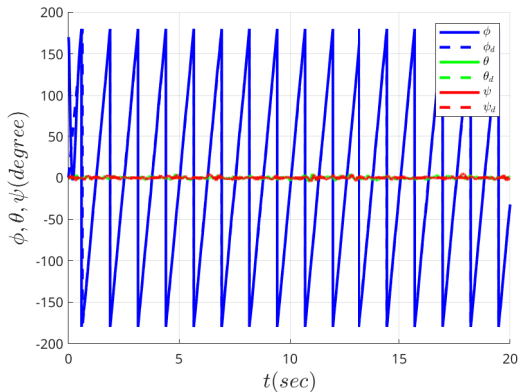


Fig. 9. Euler angles tracking the desired signal in the presence of noise

Figure 7 and Figure 8 show the attitude and body rate errors converging to zero in the presence of noise. We conclude this section by showing performance of the proposed controller while performing multiple flips in Figure 9.

VI. CONCLUSIONS

In this paper, we have solved the classical geometric tracking problem for a fully-actuated rigid body and gave both local and almost-global solutions for rigid body tracking in a coordinate-free way. To solve the problem, we defined a

novel family \mathcal{F}_R of Lie algebra valued functions on $\text{SO}(3)$. We then proposed a procedure of designing controllers from the \mathcal{F}_R family. The controllers induced by the \mathcal{F}_R family form a controller class \mathcal{C}_R . Asymptotic stability of the whole controller class \mathcal{C}_R is proven. The controller class \mathcal{C}_R consists of both local and almost global geometric controllers. Since it is impossible to design a smooth continuous globally converging controller, we underscore that in terms of region of convergence, this \mathcal{C}_R controller class contains the strongest possible controllers, i.e., almost-globally converging controllers.

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VII. APPENDIX

In this section, we present some elementary closed-form expressions of derivatives of functions involving rotation matrices. Given $R, R_d \in \text{SO}(3)$, and $\widehat{\Omega}, \widehat{\Omega}_d \in \mathfrak{so}(3)$, satisfying $\dot{R}(t) = R(t)\widehat{\Omega}(t)$, and $\dot{R}_d(t) = R_d(t)\widehat{\Omega}_d(t)$, then direct calculations lead to the following:

$$\begin{aligned} \frac{d}{dt}(R_d(t)^\top R(t)) &= (R_d(t)^\top R(t))\widehat{\Omega}(t) \\ &\quad - \widehat{\Omega}_d(t)(R_d(t)^\top R(t)); \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}(R(t)^\top R_d(t)) &= (R(t)^\top R_d(t))\widehat{\Omega}_d(t) \\ &\quad - \widehat{\Omega}(t)(R(t)^\top R_d(t)). \end{aligned}$$

Next, we present an elementary but useful result.

Proposition VII.1. *Every matrix 1-parameter subgroup of $\text{SO}(3)$ is of form $R(t) = \exp(t\widehat{\Omega})$. Moreover,*

$$\frac{d}{dt}(\text{Log}(R(t))) = \widehat{\Omega}.$$

Proof. By [18, Proposition 5.7], $\exp : U_{\mathfrak{so}(3)} \rightarrow U_{\text{SO}(3)}$ is an analytic diffeomorphism, and by [18, Lemma 5.17], every matrix 1-parameter subgroup of $\text{SO}(3)$ is of form $R(t) = \exp(t\widehat{\Omega})$. For every 1-parameter subgroup of $\text{SO}(3)$ and for every fixed $\widehat{\Omega} \in U_{\mathfrak{so}(3)}$, we can write

$$R(t) = \exp(t\widehat{\Omega}). \quad (21)$$

By taking Log of both sides, we get

$$\begin{aligned} \text{Log}(R(t)) &= \text{Log}(\exp(t\widehat{\Omega})) \\ &= t\widehat{\Omega}. \end{aligned} \quad (22)$$

By taking the derivative of the above expression, we obtain

$$\frac{d}{dt}(\text{Log}(R(t))) = \widehat{\Omega}. \quad (23)$$

□

Given $R, R_d \in \text{SO}(3)$ and $\widehat{\Omega}, \widehat{\Omega}_d \in \mathfrak{so}(3)$, and the dynamics given by $\dot{R}(t) = R(t)\widehat{\Omega}(t)$ and $\dot{R}_d(t) = R_d(t)\widehat{\Omega}_d(t)$, then direction calculations yield the following expressions:

$$\begin{aligned} \frac{d}{dt}(\text{Log}(R_d(t)^\top R(t))) &= \widehat{\Omega}(t) \\ &\quad - (R(t)^\top R_d(t))\widehat{\Omega}_d(t)(R_d(t)^\top R(t)); \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}(\text{Log}(R(t)^\top R_d(t))) &= \widehat{\Omega}_d(t) \\ &\quad - (R_d(t)^\top R(t))\widehat{\Omega}(t)(R(t)^\top R_d(t)); \end{aligned}$$

$$\frac{d}{dt}(\text{Log}(R(t)R_d(t)^\top)) = \text{Ad}_{R_d(t)}\widehat{\Omega}(t) - \text{Ad}_{R_d(t)}\widehat{\Omega}_d(t);$$

$$\frac{d}{dt}(\text{Log}(R_d(t)R(t)^\top)) = \text{Ad}_{R(t)}\widehat{\Omega}_d(t) - \text{Ad}_{R(t)}\widehat{\Omega}.$$

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