Hybrid Geometric Controllers for Fully-Actuated Left-invariant Systems on Matrix Lie Groups

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***Abstract*— This paper proposes a hybrid geometric control scheme for a system defined on a matrix Lie group in the form of a left-invariant vector field. Our solution to the point stabilization problem is coordinate free (or geometric). Specifically, we propose a hybrid geometric controller that uses a controller from a local class of geometric controllers and an open-loop geometric controller. Our method guarantees that the given point in the manifold is robustly globally asymptotically stable for the closed-loop system when each controller from the local geometric class is combined with the geometric open-loop controller using a hybrid systems framework.**

I. Introduction

A large class of mechanical, robotic, and physical systems evolve on a Lie group [1]–[3]. Intuitively speaking, the state space of these geometric systems are "curved" spaces; hence, classical analysis and controls tools are not directly applicable [1], [4]. Since a curved surface *locally* looks like a flat surface, this idea can be exploited, and one can construct a local diffeomorphism (local-coordinate chart) from an open subset of that curved state space into an open set of $\mathbb{R}^n$. However, as the name suggests, any controller developed using a local-coordinate chart would be local and suffers singularities. This paper designs a global controller for a class of systems evolving on Lie groups in a geometric or coordinate-free setting.

The work in [5], [6] considers systems on Lie groups and built geometric foundations of the left-invariant control system. However, global point stabilization on a compact manifold using a smooth controller is a nontrivial problem. In fact, it is impossible to design a global smooth feedback controller that stabilizes a point on a compact manifold. The difficulty arises because of topological obstructions, which can be characterized in terms of critical points by using Morse functions [7]. Reeb's theorem [7] implies that even in a simple Lie group such as the unit sphere, a Morse function has two critical points. It is easy to visualize that if the system is initialized on one critical point, it can never reach the second critical point using any smooth control scheme. Hence, global point stabilization using a smooth controller on a Lie group fails even when the state space is as simple as a unit circle. However, a hybrid control scheme can be used to achieve global results [8], [9].

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In recent literature, hybrid controllers that are geometric, or coordinate free, are developed for the stabilization of systems on specific manifolds or Lie groups [10]–[12]. Moreover, in [13], [14], the authors design a geometric hybrid controller for systems on $\mathsf{SO}(3)$ and $\mathsf{SE}(3)$. However, one major limitation of the work mentioned above is that the results are only applicable to a specific Lie group or a relatively smaller class of Lie groups. The results presented in this paper are more general as we consider a large class of both compact and noncompact matrix Lie groups and design robust global hybrid geometric controllers.

This paper designs a geometric or coordinate-free controller for a class of systems defined on matrix Lie groups. Our proposed method guarantees robust global point stabilization of left-invariant systems defined on both compact and noncompact matrix Lie group. Our controller design is a three-step process. In the first step, we design a local class of geometric controllers that provides asymptotic stability in a neighbourhood of the desired point. An open-loop geometric controller forces the system's trajectory to enter the local neighbourhood defined in the first step. Finally, in the third step, a switching mechanism is designed by introducing hysteresis and using tools of hybrid control to guarantee robust global stability of the desired point. Due to space constraints, proofs of the results will be published elsewhere.

The main contributions of this paper are as follows:

1) A novel kinematic family of Lie algebra valued function $\mathcal{F}_k$ on $\mathbb{S}^1$ (Definition 5.3);
2) A class of geometric kinematic controllers $\mathcal{C}_k$ that provides asymptotic stability (Lemma 5.5);
3) A class of hybrid controllers that guarantee robust global asymptotic stability to the desired point (Theorem 5.12).

A. Notation and Math Preliminaries

Let $\mathbb{N}, \mathbb{R}$, and $\mathbb{Z}$ be the set of natural numbers, real numbers, and integers, respectively. The n-dimensional Euclidean space is represented by $\mathbb{R}^n$. For a point $x \in \mathbb{R}^n$, the Euclidean norm is denoted by $|x|$, and the distance of a point x from a subset $S \subset \mathbb{R}^n$ is represented by $|x|_S := \inf_{y \in S} |x - y|$. The closure of a set S is denoted by $\overline{S}$. The closed unit ball of appropriate dimension in Euclidean norm is denoted by $\mathbb{B}$. For a matrix $A \in \mathbb{R}^{m \times n}$, its Frobenius norm is given by $\|A\|_F$. We denote the inner product of two vectors $x, y \in \mathbb{R}^n$ as $\langle x, y \rangle$. A k-dimensional vector x is represented as $(x_1, x_2, \ldots, x_k) := \begin{bmatrix} x_1, & x_2, & \cdots & x_k \end{bmatrix}^\top$, where $^\top$ denotes transposition. The domain of a map f is represented by $\operatorname{dom} f$. The value of the gradient of the map $f : \mathbb{R}^m \to$

$\mathbb{R}^n$ with respect to its argument evaluated at x is given by $\nabla f(x)$. The trace and determinant of a matrix $A \in \mathbb{R}^{n\times n}$ is represented by $\mathrm{trace}(A)$ and $\det(A)$, respectively. The set of all $n \times n$ real invertible matrices are denoted by $\mathsf{GL}(n,\mathbb{R})$ The set of $n\times n$ rotation matrices is defined as $\mathsf{SO}(n) = \left\{R \in \mathbb{R}^{n\times n} : R^\top = R^{-1}, \det(R) = +1\right\}$ and it has a Lie group structure. The associated Lie algebra of $\mathsf{SO}(n)$ is the set of $n\times n$ skew-symmetric matrices $\mathfrak{so}(n) = \left\{A \in \mathbb{R}^{n\times n} : A = -A^\top\right\}$, which form a vector space. One can construct a diffeomorphism between Euclidean space and Lie algebra by $\widehat{\cdot} : \mathbb{R}^n \to \mathfrak{g}$ and its inverse map is given by $(\cdot)^\vee : \mathfrak{g} \to \mathbb{R}^n$. The unit n-sphere is defined as $\mathbb{S}^n := \left\{x \in \mathbb{R}^{n+1} : |x| = 1\right\}$. It is well known that $\mathbb{S}^1$ is a Lie group and is isomorphic to $\mathsf{SO}(2)$.

Next, we present briefly the mathematical machinery needed for this paper; for a complete insight, the readers are referred to [1], [15]. Given two smooth manifolds M, and N, let $f : \mathsf{M} \to \mathsf{N}$ be a smooth map. For $x \in \mathsf{M}$, the tangent space of M at point x is denoted by $\mathsf{T}_x\mathsf{M}$. Let TM be the tangent bundle of the manifold M consisting of all disjoint union of tangent spaces of M.

Definition 1.1 (Vector field): A *vector field* X on a manifold M is a map $X : \mathsf{M} \to \mathsf{TM}$ that assigns a vector $X(p)$ at a point $p \in \mathsf{M}$ with the restriction that $\pi_{\mathsf{TM}} \circ X = \mathsf{id}$, where π_{TM} is the tangent bundle projection map.

The set of all smooth vector fields on a manifold M is denoted by $\mathfrak{X}(\mathsf{M})$. Vector fields represent differential equations on manifolds [15]. The set of all $n \times n$ real invertible matrices form a Lie group and is called the general linear group $\mathsf{GL}(n,\mathbb{R})$. A *matrix Lie group* is a subgroup of $\mathsf{GL}(n,\mathbb{R})$. The space of all real $n\times n$ matrices (both invertible and non-invertible) with the matrix commutator $[A,B] := AB - BA$ is a Lie algebra, called the general Lie algebra $\mathfrak{gl}(n,\mathbb{R})$.

Definition 1.2 (Left-invariant control systems): A left-invariant control system on a real, matrix Lie group G with m-inputs consists of a family of left-invariant vector fields $X \in \mathfrak{X}(\mathsf{TG})$ on the tangent bundle. Such a system can be written as

$$\dot{g} = X(g,u) = g\left(A + \sum_{i=1}^{m} B_i u_i\right) = g\xi, \tag{1}$$

where $g \in \mathsf{G}$, $\xi := \left(A + \sum_{i=1}^{m} B_i u_i\right) \in \mathfrak{g}$, and $(u_1, u_2, \ldots, u_m) \in \mathbb{R}^m$. The element A belongs to the associated Lie algebra $\mathfrak{g}$ over the field $\mathbb{F}$, the elements $B_1, B_2, \ldots, B_m$ form a basis of the Lie algebra. If A is zero the system is called driftless because $\dot{g} = 0$ whenever $u_i = 0$ for all $i \in \{1,2,\ldots,m\}$. Moreover, the system is fully actuated if $B_i \neq 0$ for all $i \in \{1,2,\ldots,m\}$.

Remark 1.3: The tangent bundle of TG is trivializable, i.e., $\mathsf{TG} \cong \mathsf{G} \times \mathfrak{g}$.

Next, we state a useful result from Morse theory [7].

Corollary 1.4: Nondegenerate critical points of Morse functions (smooth, real-valued functions) on smooth manifolds are isolated. Moreover, a Morse function on a compact manifold has finitely many critical points, which are all isolated.

II. MOTIVATIONAL EXAMPLES

Before providing a global control law for the left-invariant system (1), we provide two examples of fully actuated systems evolving on Lie groups.

Example 2.1: Consider the set of three-by-three upper triangular matrices of the form

$$\mathsf{H}(3) = \left\{(x,y,z) := \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x,y,z \in \mathbb{R}\right\}. \tag{2}$$

The set $\mathsf{H}(3)$ forms a group and is called the Heisenberg group. The group identity is defined as $e := (0,0,0)$, the group multiplication is matrix multiplication and the group inverse is defined for an element (x,y,z) as $(x,y,z)^{-1} := (-x,-y,xy-z)$. Thus, $\mathsf{H}(3)$ forms a group under group (matrix) multiplication and it is a Lie group. The Heisenberg algebra of $\mathsf{H}(3)$ is given by

$$\mathfrak{h}(3) = \left\{\begin{bmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix} : a,b,c \in \mathbb{R}\right\}.$$

We can pick the following basis vectors for the Heisenberg algebra $\mathfrak{h}(3)$:

$$B_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, B_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

A freely-moving, fully-actuated particle in the three-dimensional space can be modelled as a system evolving on the Heisenberg group $\mathsf{H}(3)$ and is given in the form (1) as

$$\dot{H} = H\left(B_1u_1 + B_2u_2 + B_3u_3\right), \tag{3}$$

where u_1, u_2, u_3 are the scalar control inputs and $H \in \mathsf{H}(3)$.

Example 2.2: Consider a fully-actuated planar robot. Its configuration is defined on Special Euclidean group

$$\mathsf{SE}(2) = \left\{E \in \mathbb{R}^{3\times3} : E = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}, R \in \mathsf{SO}(2), p \in \mathbb{R}^2\right\}.$$

The set $\mathsf{SE}(2)$ forms a group under the following multiplication and inverse rule:

$$E_1 \cdot E_2 := \begin{bmatrix} R_1 & p_1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} R_2 & p_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1R_2 & R_1p_2 + P_1 \\ 0 & 1 \end{bmatrix},$$

and

$$E_1^{-1} := \begin{bmatrix} R_1^\top & -R_1^\top p_1 \\ 0 & 1 \end{bmatrix},$$

for $E_1, E_2 \in \mathsf{SE}(2)$ and the identity element is the three-by-three identity matrix. Its Lie algebra is given by

$$\mathfrak{se}(2) = \left\{W \in \mathbb{R}^{3\times3} : W = \begin{bmatrix} \omega & p \\ 0 & 1 \end{bmatrix}, \omega \in \mathfrak{so}(2), p \in \mathbb{R}^2\right\}.$$

For some basis vectors $B_1, B_2, B_3 \in \mathfrak{se}(2)$ the kinematics of a planar rigid body can be expressed in the form (1) as

$$\dot{E} = E\left(B_1 u_1 + B_2 u_2 + B_3 u_3\right), \tag{4}$$

where u_1, u_2, and u_3 are the scalar control inputs.

III. Hybrid Systems on Manifolds

Informally, a hybrid control system consists of a hybrid plant and a hybrid controller whose variables may evolve continuously, called *flow*, or change instantaneously, called *jump*. We refer the reader to [8], [16] for more details. First, we provide the notion of hybrid time.

Definition 3.1 (hybrid time and hybrid time domain): Hybrid time is defined by pairs (t, j), where $t \in \mathbb{R}_{\geq 0}$ captures the duration of flows and $j \in \mathbb{N}$ indicates the number of jumps. A set E is a *hybrid time domain* if for all $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, ..., J\})$ is a compact hybrid time domain; i.e., it can be written as $\cup_{j=0}^{J-1}([t_j, t_{j+1}], j)$ for some finite sequence of times $0 \leq t_0 \leq t_1 \leq \ldots \leq t_J$.

Definition 3.2 (hybrid plant): A hybrid equation model of a plant with hybrid dynamics on a manifold M_P is given by

$$\mathcal{H}_P : \begin{cases} (z, u) \in C_P & \dot{z} \;= F_P(z, u) \\ (z, u) \in D_P & z^+ = G_P(z, u) \\ & y \;= h(z), \end{cases} \tag{5}$$

where the state z takes values from the manifold, i.e., $z \in \mathsf{M}_P$, the inputs to the plant take values from a subset of the Euclidean space, i.e., $u \in \mathcal{U}_P \subset \mathbb{R}^{m_p}$, and y is the output of the plant. Moreover, the set $C_P \subset \mathsf{M}_P \times \mathcal{U}_P$ is called the *flow set*, the set $D_P \subset \mathsf{M}_P \times \mathcal{U}_P$ is called the *jump set*, the single-valued mapping F_P: $\mathsf{M}_P \times \mathcal{U}_P \rightarrow \mathsf{TM}_P$ is called the *flow map*, and the single-valued mapping $G_P : \mathsf{M}_P \times \mathcal{U}_P \rightarrow \mathsf{M}_P$ is called the *jump map*. The data of the hybrid plant is defined by the tuple (C_P, F_P, D_P, G_P, h).

Unlike [8], the states of the plant evolve on the smooth manifold M_P. It should be noted that (1) is a special case of the hybrid plant $\mathcal{H}_P$ because the system only flows, i.e., $D_P = \emptyset$. In fact, in this particular case, the state and input of the system are $z := g \in \mathsf{G} = \mathsf{M}_P$ and $\xi \in \mathfrak{g} = \mathcal{U}_P$, respectively. Moreover, the data (C_P, F_P, D_P, G_P) is given as $C_P := \mathsf{TG}$, $F_P(g, u) := g\xi$, $D_P = \emptyset$, and G_P can be any arbitrary mapping. This definition captures a continuous-time plant evolving on a manifold; for more details, see [8]. Similarly, a hybrid controller model can be defined as follows.

Definition 3.3 (hybrid controller): A hybrid equation model of a controller with hybrid dynamics is given by

$$\mathcal{H}_K : \begin{cases} (v, \eta) \in C_K & \dot{\eta} \;= F_K(v, \eta) \\ (v, \eta) \in D_K & \eta^+ = G_K(v, \eta) \\ & \zeta \;= \kappa(v, \eta), \end{cases} \tag{6}$$

where η is the state, v is the input, and ζ is the output of the controller. Moreover, C_K is the flow set, D_K is the jump set, F_K is the flow map, and G_K is the jump map. The data of the hybrid controller is defined by the tuple $(C_K, F_K, D_K, G_K, \kappa)$.

The control of the plant $\mathcal{H}_P$ via the controller $\mathcal{H}_K$ defines an interconnection through the following rule: $u = \zeta$ and $v = y$. Similar to the hybrid plant and the hybrid controller, a hybrid closed-loop system can be defined as follows.

Definition 3.4 (hybrid closed-loop system): A hybrid equation model of the closed-loop system on a manifold M is given by

$$\mathcal{H} : \begin{cases} x \in C & \dot{x} \;= F(x) \\ x \in D & x^+ = G(x) \end{cases} \tag{7}$$

where x is the state evolving on the manifold M, C is the flow set, D is the jump set, $F : \mathsf{M} \rightarrow \mathsf{TM}$ is the flow map, and $G : \mathsf{M} \rightarrow \mathsf{M}$ is the jump map. The data of the hybrid closed-loop system is defined by the tuple (C, F, D, G).

IV. Problem Formulation

As mentioned earlier, global asymptotic stabilization of fully-actuated systems on Lie groups G of the form (1) is a nontrivial problem. This paper shows that a hybrid controller can be designed to achieve robust global asymptotic stabilization for a continuous-time system defined on both compact and noncompact Lie groups, even in the presence of noise. Without loss of generality, let $e \in \mathsf{G}$ be the point we want to stabilize, where e is the identity element of the Lie group.

Problem 1: Given a fully actuated left-invariant system on a finite dimensional real matrix Lie group G given in the form (1), design a controller such that each maximal solution component $(t, j) \mapsto g(t, j)$ to the closed-loop system is complete and globally asymptotically converges to the desired point $e \in \mathsf{G}$, i.e., for all $g(0, 0) \in \mathsf{G}$

$$\lim_{t+j \rightarrow \infty} g(t, j) = e,$$

with stability and robustness to small perturbations.

In the next section, we characterize the class of controllers that solves this problem.

V. Controller Design

To overcome the topological obstruction and to achieve global and robust asymptotic stabilization on Lie groups, we design a hybrid control algorithm that properly unites two controllers. Next, we introduce a few definitions and auxiliary results. First, we denote the map that measures the distance between two elements in G as $d_{\mathsf{G}} : \mathsf{G} \times \mathsf{G} \rightarrow \mathbb{R}$. Since every Lie group is also a Riemannian manifold, it is always possible to find such a map.

Definition 5.1 (closed-ball): Given a point $g^\star \in \mathsf{G}$, the unit closed ball in G around $g^\star \in \mathsf{G}$ is defined as $\mathbb{B}_{g^\star}^{\mathsf{G}} := \{g \in \mathsf{G} : d_{\mathsf{G}}(g, g^\star) \leq 1\}$.

Definition 5.2: Given a point $g^\star \in \mathsf{G}$ and $\epsilon > 0$, the open ϵ-neighborhood of $g^\star$ is defined as $\mathcal{N}_\epsilon(g^\star) := \{g \in \mathsf{G} : d_{\mathsf{G}}(g, g^\star) < \epsilon\}$.

It should be noted that the ϵ-neighbourhood of every point in G is a set of nonzero Lebesgue measure. We make the following assumption on G.

Assumption 1: Let G be a connected matrix Lie group. Every critical point of a smooth real-valued function is nondegenerate.
It follows from Corollary 1.4 that for a Morse function on G, each critical point is isolated and, hence, nondegenerate. It is easy to see that every manifold has at least one critical point. We define the set of (not necessarily finite) isolated critical points by

$$\mathsf{C}_{rt} := \{c_0^\star, c_1^\star, c_2^\star, c_3^\star, \cdots\},$$

where $c_i^\star \in \mathsf{G}$ for $i \in \{0, 1, 2, \dots\}$ is a nondegenerate critical point. Without loss of generality, we assume that $c_0^\star = e$. Since G is Hausdorff, this ensures the existence of $\epsilon > 0$ and an open ϵ-neighbourhood of $e \in \mathsf{G}$ that does not contain any other critical point, i.e., $\mathcal{N}_\epsilon(e) \cap \mathcal{N}_\epsilon(c_i^\star) = \emptyset$ for all $i \in \{1, 2, \dots\}$.

A. A Family of Geometric State-Feedback Controllers

We design a class of geometric controllers on G that will be used locally, nearby the point e. We propose a novel family of Lie algebra valued functions on G.

Definition 5.3: A function $f : \mathbb{D} \subset \mathsf{G} \to \mathfrak{g}$ is said to belong to the kinematic family of Lie algebra valued functions $\mathcal{F}_k$ if it satisfies the following properties:

1) f is at least C^1;
2) $f^{-1}(0) = \{g \in \mathbb{D} : g = e\}$;
3) $\mathrm{d}_g f$, the derivative of f with respect to g, is nonsingular at least in a neighbourhood of e;
4) $\mathbb{D}$ contains an open neighborhood of e and is connected.

Remark 5.4: On every Lie group, one can define $\log$ and $\exp$ maps. Moreover, for every Lie group $\log$ is defined at least on $\mathcal{N}_\epsilon(e)$. Therefore, the existence of functions in the family $\mathcal{F}_k$ is guaranteed. It should be noted that there may exist functions other than $\log$. For example, on $\mathsf{SO}(3)$ an example of a function that belongs to $\mathcal{F}_k$ is $f(g) = g - g^{-1}$ defined for each $g \in \mathsf{SO}(3)$.

Lemma 5.5: Given a left-invariant fully-actuated system as in (1) defined on a matrix Lie group G with the identity element $e \in \mathsf{G}$, and a kinematic family of Lie algebra valued functions $\mathcal{F}_k$, each function $f \in \mathcal{F}_k$ induces a controller given by

$$\sum_{i=1}^{m} B_i u_i = \kappa(g) = g^{-1} \left(\mathrm{d}_g f\right)^{-1} (-f(g)), \tag{8}$$

such that for the resulting closed-loop system obtained from controlling (1), namely,

$$\dot{g} = g\kappa(g), \tag{9}$$

the singleton set $\{g \in \mathbb{D} : g = e\}$ is asymptotically stable with basin of attraction given by

$$\mathcal{B}_f := \{g \in \mathbb{D} : \det(\mathrm{d}_g f(g)) \neq 0\}. \tag{10}$$

Remark 5.6: Every function f contained in the kinematic family $\mathcal{F}_k$ gives rise to a local geometric controller, which, by Lemma 5.5, renders $e \in \mathsf{G}$ asymptotically stable with the basin of attraction $\mathcal{B}_f$ in (10). The collection of all such controllers constitutes a class, denoted by $\mathcal{C}_k$, that we call the *kinematic controller class*. When the underlying Lie group is a noncompact Lie group, such as a Heisenberg group $\mathsf{H}(3)$, there exist functions in the family $\mathcal{F}_k$ that lead to controllers whose domain of attraction might be the whole Lie group.

Example 5.7: Continuing from Example 2.1, we select the following function from the $\mathcal{F}_k$ family:

$$f : \mathsf{H}(3) \to \mathfrak{h}(3), \quad H \mapsto \log(H).$$

In light of Lemma 5.5, it is straightforward to verify that $\mathrm{d}_H f = e$. Therefore, f is invertible everywhere and produces the following controller:

$$\sum_{i=1}^{m} B_i u_i = \kappa_0(H) = -\log(H). \tag{11}$$

The controller κ_0 is defined everywhere on $\mathsf{H}(3)$. In other words, the controller κ_0 has a basin of attraction equal to the state space, i.e., $\mathcal{B}_f = \mathsf{H}$, hence is globally asymptotically stabilizing.

Example 5.8: Continuing from Example 2.2, let $I \in \mathsf{SE}(2)$ be the identity element and $Z := \{E \in \mathsf{SE}(2) : R \neq -I\}$. We select the following function from the $\mathcal{F}_k$ family:

$$f : \mathsf{SE}(2) \setminus Z \to \mathfrak{se}(2), \quad E \mapsto \log(E).$$

Again $\mathrm{d}_E f = I$. Therefore, f is invertible everywhere and produces the following controller:

$$\sum_{i=1}^{m} B_i u_i = \kappa_0(E) = -\log(E). \tag{12}$$

The controller κ_0 is defined everywhere on $\mathsf{SE}(2)$ except for the set of measure zero Z. In other words, the controller κ_0 has the largest possible basin of attraction, i.e., $\mathcal{B}_f = \mathsf{SE}(2) \setminus Z$. This controller is almost globally asymptotically stabilizing.

For other candidate functions from the family $\mathcal{F}_k$, the domain of attraction can be smaller than the one considered in Example 5.8. However, it must be noted that Definition 5.3 and Lemma 5.5 guarantee that the region of attraction will be a nonempty open neighbourhood of $e \in \mathsf{G}$.

B. A Geometric Open-Loop Controller

In this section, we define a local open-loop controller that, from initial conditions in a particular initial set, is able to steer the state to points in the interior of the basin of attraction of a controller in the kinematic controller class $\mathcal{C}_k$. To define this controller, we need the following assumption.

Assumption 2: For each $i \in \{1, 2, \dots\}$, there exists an ϵ_i-neighbourhood of $c_i^\star$ and $b_i > 1$, such that the sets

$$\tilde{C}_1 := \bigcup_{i=1}^{\infty} \mathcal{N}_{\epsilon_i}(c_i^\star), \quad C_1 := \bigcup_{i=1}^{\infty} \mathcal{N}_{b_i \epsilon_i}(c_i^\star)$$

are connected, $\tilde{C}_1 \cap \mathcal{N}_{\epsilon_0}(e) = \emptyset$, and $C_1 \cap \mathcal{N}_{\epsilon_0}(e) = \emptyset$.

Lemma 5.9: Let C_1 and $\tilde{C}_1$ be defined in Assumption 2, and define $C_0 := \mathsf{G} \setminus \tilde{C}_1$. Let $\{B_1, B_2, \dots, B_m\}$ be the basis of the Lie algebra $\mathfrak{g}$. For each $g(0) \in C_1$, the open-loop controller

$$\kappa_1(g) = \sum_{i=1}^{m} B_i u_i \tag{13}$$

with $u_i = 1$ for each $i \in \{1, 2, \ldots, m\}$ is such that the solution $t \mapsto g(t)$ to system (1) under the effect of κ_1 reaches in finite time a nonempty set, denoted $\mathcal{T}_{1,0}$, that contains e and that is a subset of C_0, i.e., for each $g(0) \in \mathsf{G}$, there exists $T > 0$ such that the solution $t \mapsto g(t)$ satisfies $g(T) \in \mathcal{T}_{1,0} \subset C_0$.

Remark 5.10: Note that $\tilde{C}_1 \cap C_1 \neq \emptyset$ and $\tilde{C}_1 \subset C_1$, and, by construction, $C_0 \cap C_1 \neq \emptyset$. Furthermore, the critical point $e \in \mathsf{C}_{rt}$ is in the interior of C_0.

C. Hybrid Control Algorithm

The construction of the controller class $\mathcal{C}_k$ is such that a controller κ_0 in that class asymptotically stabilizes the critical point e with basin of attraction $\mathcal{B}_f$ defined in (10). If the initial state $g(0)$ is not in the basin of attraction, we propose to use the open-loop controller κ_1 given in (13) in Lemma 5.9 to steer g into $\mathcal{B}_f$. At first glance, it may appear that a discontinuous (non-hybrid) switching scheme would be sufficient to achieve robust and global stabilization. Nevertheless, such a solution would be sensitive to even arbitrarily small noise and, hence, nonrobust [8]. In other words, in the presence of noise, some solutions of the system may exhibit chattering at the switching surface when a discontinuous controller is used [8], [10].

To guarantee robustness and prevent chattering at switching, we propose a hybrid controller $\mathcal{H}_K$ as in (6) that properly orchestrates the use of κ_0 and κ_1. To this end, let C_1 and $\tilde{C}_1$ be defined in Assumption 2. Moreover, let C_0 and $\mathcal{T}_{1,0}$ come from Lemma 5.9. We assume the following property for these sets and the controller κ_0.

Assumption 3: Every maximal[1] solution to (1) under the effect of κ_0 that starts from a closed set $\mathcal{T}_{1,0}$ is complete[2] remains in the interior of C_0 for all time.

Remark 5.11: Completeness of maximal solutions to (1) under the effect of κ_0 can be established by regularity properties of κ_0 (e.g., Lipschitzness on G). The property of solutions from $\mathcal{T}_{1,0}$ staying in the interior of C_0 is a conditional invariance property that can be guaranteed using the results in [17].

With these objects and properties, the hybrid controller $\mathcal{H}_K$ has state $\eta = q \in Q := \{0, 1\}$, input $v = z := g \in \mathsf{G}$, output $\zeta := \xi \in \mathfrak{g}$, and data $(C_K, F_K, D_K, G_K, \kappa)$ as follows:

$$C_K = \bigcup_{q \in Q} (C_{K,q} \times \{q\}), \quad \begin{cases} C_{K,0} := \overline{C_0} \\ C_{K,1} := \overline{\mathsf{G} \setminus \mathcal{T}_{1,0}} \end{cases} \tag{14}$$

$$F_K(z, q) = 0 \quad \forall (z, q) \in C_K \tag{15}$$

[1]A maximal solution is a solution that cannot be further extended.

[2]A complete solution to (1) has a domain of definition that is equal to $[0, \infty)$.

$$D_K = \bigcup_{q \in Q} (D_{K,q} \times \{q\}), \quad \begin{cases} D_{K,0} := \overline{\mathsf{G} \setminus C_0} \\ D_{K,1} := \mathcal{T}_{1,0} \end{cases} \tag{16}$$

$$G_K(z, q) = 1 - q \quad \forall (z, q) \in D_K \tag{17}$$

$$\kappa(z, q) = q\kappa_1(z) + (1 - q)\kappa_0(z). \tag{18}$$

Controlling the continuous-time plant (1), defined on a Lie group, via the hybrid controller results in a hybrid closed-loop system on a manifold with state $x = (z, q) \in \mathsf{G} \times Q$ and dynamics

$$\dot{z} = F_P(z, \kappa(z, q)) := g\kappa(z, q), \quad \dot{q} = 0 \tag{19}$$

during flows, and at jumps, the state is updated according to

$$z^+ = z, \quad q^+ = 1 - q. \tag{20}$$

Finally, the hybrid closed-loop system $\mathcal{H} = (C, F, D, G)$ with the state $x = (z, q) \in \mathsf{G} \times Q =: \mathcal{X}$ has data given as

$$\begin{aligned} C &:= \{(z, q) \in \mathcal{X} : (z, \kappa(z, q)) \in C_P, z \in C_{K,q}\} \\ F(x) &:= \begin{bmatrix} F_P(z, \kappa(z, q)) \\ 0 \end{bmatrix} \quad \forall x \in C \\ D &:= \{(z, q) \in \mathcal{X} : (z, \kappa(z, q)) \in C_P, z \in D_{K,q}\} \\ G(x) &:= \begin{bmatrix} z \\ 1 - q \end{bmatrix} \quad \forall x \in D, \end{aligned} \tag{21}$$

where $C_P := \mathsf{G}$. Our main result is as follows.

Theorem 5.12: Given the critical point $e \in \mathsf{G}$ and the continuous-time plant (1) defined on a Lie group G, suppose Assumptions 1-3 hold. Let the hybrid controller $\mathcal{H}_K$ with data $(C_K, F_K, D_K, G_K, \kappa)$ be defined in (14)-(18). The following hold:

1) The closed-loop system $\mathcal{H} = (C, F, D, G)$ with data in (21) satisfies the hybrid basic conditions;
2) Every maximal solution to $\mathcal{H}$ from $C \cup D$ is complete and exhibits no more than two jumps;
3) The set $\mathcal{A} = \{e\} \times \{0\}$ is robustly globally asymptotically stable for $\mathcal{H}$ in the sense of Definition 8.1.

VI. Simulation Results

In this section, we provide simulation results of the hybrid controller $\mathcal{H}_K$ designed for the system evolving on $\mathsf{SE}(2)$ and presented in Examples 2.2 and 5.8. Informally, we unite geometric controller κ_0 and the open-loop controller κ_1, given in (12) and (13), respectively, through the hybrid framework. The system is initialized at position $x(0, 0) = 10$ m and $y(0, 0) = -5$ m, while the orientation is at the most challenging position, i.e., $R(0, 0) = -I$. No controller from the $\mathcal{C}_k$ controller class can make the system states converge asymptotic to the desired point. In other words, the system is initialized on the set $C_{K,1} \times \{1\}$; therefore, the hybrid controller $\mathcal{H}_K$ selects the controller κ_1. The system trajectory flows for about 1 seconds, as seen in Figure 1(b) and then enters in the set $C_{K,0} \times \{0\}$. After that, the hybrid controller $\mathcal{H}_K$ gives the control authority to the controller κ_0, which makes the system asymptotically converge to the desired point $e \in \mathsf{SE}(2)$, as seen in Figure 1(a). It can be seen

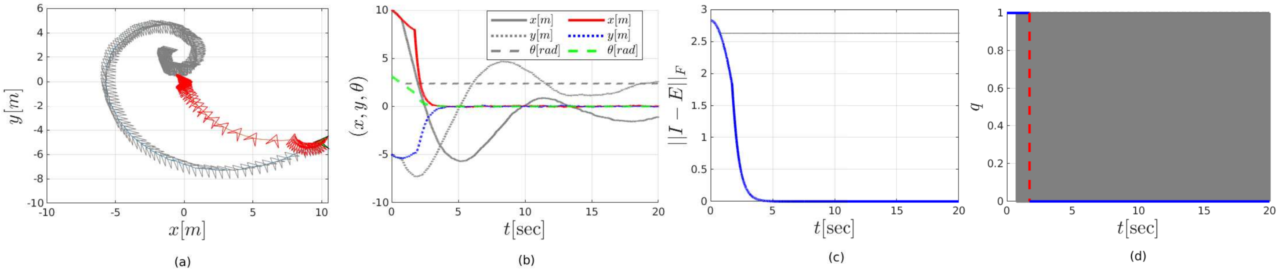

Fig. 1. A left-invariant system on $\mathsf{SE}(2)$ demonstrates global convergence to the desired pose using the hybrid control scheme. The logic variable q switches from 1 to 0 indicating the switching of the control authority. Without a hybrid (non-hysteresis) scheme, we get undesirable results as shown in grey.

in Figure 1(c) that the system pose E reaches the desired pose I. We simulate the system under persistent random white noise and demonstrate that even in the presence of noise, the system converges to the desired point. Moreover, as shown in Figure 1(d), around 1 second, the control authority switches from controller κ_1 to κ_0. Finally, we show how the system behaves without a hybrid controller scheme, i.e., in the absence of a hysteresis gap. As shown in Figure 1, in grey, without hysteresis, the system fails to achieve point stabilization, the logic variable q switches between one and zero multiple times, Figure 1(d), and the errors do not converge to zero, grey line in Figure 1(c). Without hysteresis, the position and orientation of the system also do not converge to the desired values, as shown in grey in Figure 1(b).

VII. Conclusion

We developed a hybrid geometric controller to globally robustly and asymptotically stabilize the desired point for systems defined by left-invariant vector fields on matrix Lie groups. First, we design a class of local feedback geometric controllers that guarantees the desired stability. When the system is initialized outside the basin of attraction of the feedback class of controllers, we invoke a geometric open-loop controller to force the system to enter the attraction basin. We bootstrap the geometric feedback and open-loop controller using a hybrid system framework to avoid chattering and achieve robustness.

VIII. Appendix

Definition 8.1: (robust stability [8]) Given a hybrid closed-loop system $\mathcal{H}$, a nonempty closed set $\mathcal{A} \subset \mathsf{M}$ and an open set $\mathcal{U} \subset \mathsf{M}$ such that $\mathcal{A} \subset \mathcal{U}$, the set $\mathcal{A}$ is said to be robustly stable for $\mathcal{H}$ on $\mathcal{U}$ if for every proper indicator function ϖ of $\mathcal{A}$ on $\mathcal{U}$, every function $\beta \in \mathcal{KL}$ such that

$$\varpi(x(t,j)) \leq \beta(\varpi(x(0,0)), t+j) \quad \forall (t,j) \in \operatorname{dom} x$$

for the solutions to $\mathcal{H}$ from $\mathcal{U}$, and every continuous function $\rho^* : \mathsf{M} \to \mathbb{R}_{\geq 0}$ that is positive on $\mathcal{U} \setminus \mathcal{A}$, the following holds: for each compact set $K \subset \mathcal{U}$ and each $\epsilon > 0$, there exists $\delta^* > 0$ such that for each solution x_ρ the perturbed system $\mathcal{H}_\rho$ with $\rho = \delta^* \rho^*$, starting from $x_\rho(0,0) \in K$ satisfies

$$\varpi(x_\rho(t,j)) \leq \beta(\varpi(x_\rho(0,0)), t+j) + \epsilon \quad \forall (t,j) \in \operatorname{dom} x_\rho.$$

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