Feedback Linearizing Controllers on $\mathsf{SO}(3)$ using a Global Parametrization

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***Abstract*— We present a methodology for studying the stabilization problem of a fully-actuated rotating rigid body. Since a rigid body attitude is represented by a rotation matrix in three dimensions, we exploit this fact and use each element of the rotation matrix as a parameter. This nine-parameter representation is global as well as unique, and results in a simplified set of nonlinear differential equations. We apply feedback linearization to design both local and almost global controllers. We also propose two novel definitions of feedback linearization functions, and prove that they lead to a well-defined vector relative degree and, as a result, almost-globally and locally stable controllers with bounded internal states. Using the proposed methodology, we present detailed examples of two such functions, demonstrating stabilization performance for each resulting controller on a rigid body system.**

I. INTRODUCTION

Any rigid body moving in three-dimensional space can be expressed as a rotation followed by a translation. Rigid body motion is of central importance in the field of mobile robotics as the robot needs to operate and move around in the three-dimensional space [1]. Therefore, in this work, we focus on the rigid body motion problem. Specifically, we consider the case when the translation component of the rigid body is zero, and the rigid body only undergoes rotation about its center of gravity. We assume that the rotating rigid body is fully-actuated, i.e., there are three control inputs capable of applying torques about each body axis.

In this work, we consider a rigid body stabilization problem and, without loss of generality, we assume that the desired rigid body orientation is the upright orientation. Informally, a rigid body stabilization problem is the following: given a rigid body, initialized "close" to the desired upright position, design a controller that drives the rigid body from the initial position to the desired position.

The rotational dynamics of a rigid body evolve on a manifold, i.e., $\mathsf{SO}(3)\times\mathbb{R}^3$. It is well known that $\mathsf{SO}(3)$ is a geometric object, in fact, a smooth three-dimensional manifold that forms a rotational group. Generally, control problems involving geometric structures can either be solved using geometric methods or using a parameterization, sometimes a local parameterization, of the underlying geometric structure. The problem of parameterizing $\mathsf{SO}(3)$ was first studied by Euler back in 1776. As $\mathsf{SO}(3)$ is a manifold of dimensional three, it is typical to employ three parameters to represent each point of $\mathsf{SO}(3)$. However, as shown by [2], it is not possible to have a global and non-singular three-dimensional parameterization for the rotational group due to topological obstructions. Therefore, any smooth controller designed using three parameters, e.g., Euler angles, results in singularities [3], [4]. However, it is possible to represent the rotational group globally without singularities with a 4-dimensional parameterization, such as quaternions, but such a parameterization is not 1-1, i.e., quaternions double cover $\mathsf{SO}(3)$. Controllers designed using quaternions lead to ambiguities and unwinding effects [1]. In [2], the authors highlighted the fact that the minimum number of parameters that suffice to represent $\mathsf{SO}(3)$ in both 1-1 and global manner is equal to five. However, a 5-dimensional parameterization results in a complicated set of coupled nonlinear differential equations [2], which make the controller design task extremely difficult and often result in computationally expensive and practically non-feasible controllers. On the other hand, geometric controllers lead to almost global results but control design and stability analysis require rather complicated tools of differential geometry [1], [5]–[9].

In this paper, we present a framework for controller design that is a substitute to both geometrical control method, which entails knowledge and tools of differential geometry, and the classical control method, which uses minimal (five or less) parameters. In other words, we seek a parameterization of $\mathsf{SO}(3)$ such that each point of the manifold can be represented both globally and uniquely such that the resulting set of differential equations are simple and then design a controller using elementary nonlinear control techniques, without going into the technicalities of differential geometry [10].

In this work, we exploit the fact that the rotational group can be *represented* by a three-by-three orthogonal matrix with determinant 1, also referred to as the rotation matrix . We pick every element of this rotation matrix as a parameter. In other words, we choose nine parameters to represent $\mathsf{SO}(3)$. This results in 9 coupled nonlinear differential equations. However, we show that these equations are very simple as compared to the dynamics in the case of 3-dimensional Euler angles parameterization, and also in the case of 4 and 5-dimensional parameterization. Therefore, the resulting controllers are easy to design and computationally very efficient. Our controller design procedure using this over-parameterization of smooth manifold results in uncontrolled stable internal dynamics. Moreover, we show that these uncontrolled dynamics remain well behaved.

A similar problem of tracking of rotating rigid body is

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considered in [8], where the author exploited the idea of embedding of $\mathsf{SO}(3)$ and extended the system from the manifold to the ambient space. Our idea, although somewhat similar, has a different motivation, i.e., our motivation originates from the representation theory where each element of $\mathsf{SO}(3)$ is represented by a rotation matrix, and we treated each element of the rotation matrix as a parameter. Moreover, we introduce a restriction map that diffeomorphically maps each point from $\mathsf{SO}(3)$ onto its image. Next, we state the main contributions of this paper:

1) We propose a methodology to solve a rigid body stabilization problem using 9 parameters.
2) We provide two definitions of feedback linearization functions: local feedback linearization function $\mathcal{L}_f^L$ and almost-global feedback linearization function $\mathcal{L}_f^A$.
3) We prove that the local and almost-global feedback linearization functions guarantee a local and almost-global well-defined vector relative degree, respectively.
4) $\mathcal{L}_f^A$ and $\mathcal{L}_f^L$ leads to almost-global and local controllers that asymptotically stabilize the rigid body to the desired position.
5) We show that all the internal states are stable.

The paper is organized as follows: After presenting notation and math preliminaries, we state the well-known dynamics of a fully-actuated rotating rigid body system, and express the system in the standard control-affine form using 9 parameters in Section II. After formulating the rigid body stabilization problem in Section III, we present two novel definitions and control design details in Section IV. We derive internal dynamics and prove their boundedness in Section V and conclude the paper with simulation results in Section VI.

A. Notation and Math Preliminaries

Let $\mathrm{col}(x_i,\ldots,x_k) := \begin{bmatrix} x_i & \cdots & x_k \end{bmatrix}^\top$, where $^\top$ denotes transpose. The n-dimensional Euclidean space is represented by $\mathbb{R}^n$, and $\|\cdot\|$ represents Euclidean norm. Given a C^1 map $f:\mathbb{R}^n\to\mathbb{R}^m$ and a point $p\in\mathbb{R}^n$, we denote $df_p := \frac{\partial f}{\partial x}p$. If $f,g:\mathbb{R}^n\to\mathbb{R}^n$ and $\lambda:\mathbb{R}^n\to\mathbb{R}$ are C^∞, the iterated Lie derivatives are defined as follows: $L_g^0\lambda := \lambda$, $L_g^k\lambda := L_g(L_g^{k-1}\lambda)$, $L_gL_f\lambda := L_g(L_f\lambda)$. The Euclidean norm of a vector $v\in\mathbb{R}^n$ is represented as $||v||$. Let 0_n represent a column vector of dimension n with all zero elements. Similarly, let $0_{n\times m}$ represent a matrix with n rows and m columns with all entries equal to zero and I_n represent an n by n identity matrix. The trace and determinant of a matrix $A\in\mathbb{R}^{n\times n}$ is represented by $\mathrm{tr}(A)$ and $\det(A)$, respectively. The orientation of a rotating rigid body in free space is expressed by rotation matrix R, which belongs to the set of Special Orthogonal matrices, which is defined as

$$\mathsf{SO}(3) = \left\{R\in\mathbb{R}^{3\times 3} : R^\top = R^{-1} \wedge \det(R) = +1\right\},$$

and has a Lie group structure. The associated Lie algebra of $\mathsf{SO}(3)$ is the set of 3×3 skew-symmetric matrices

$$\mathfrak{so}(3) = \left\{A\in\mathbb{R}^{3\times 3} : A = -A^\top\right\},$$

which is isomorphic to $\mathbb{R}^3$. The isomorphism is denoted by[1] $\widehat{\cdot} : \mathbb{R}^3 \to \mathfrak{so}(3)$ and its inverse is denoted $(\cdot)^\vee : \mathfrak{so}(3) \to \mathbb{R}^3$. Finally, the matrix exponential is an analytic diffeomorphism [10] between $U_{\mathfrak{so}(3)} := \left\{\widehat{\omega}\in\mathfrak{so}(3) : \omega\in\mathbb{R}^3, \|\omega\|_2 < \pi\right\}$ and $U_{\mathsf{SO}(3)} := \{R\in\mathsf{SO}(3) : \mathrm{tr}\,(R) \neq -1\}$. The inverse map from $U_{\mathsf{SO}(3)} \to U_{\mathfrak{so}(3)}$ is the principal matrix logarithm and is denoted by $\mathrm{Log}(\cdot)$.

II. MATHEMATICAL MODEL

Consider a rigid body that undergoes pure rotation, without loss of generality, about the center of gravity. Let $\Omega(t) := \mathrm{col}(p(t), q(t), r(t))$ denote the angular velocity of the system in the body-fixed frame. The well-known dynamics of a rotating rigid body can be expressed as

$$\dot{R}(t) = R(t)\widehat{\Omega}(t). \tag{1}$$

Let $J\in\mathbb{R}^{3\times 3}$ represent the inertia with respect to the body fixed-frame. Let $\tau := \mathrm{col}(\tau_p,\tau_q,\tau_r)$ be the input torques applied about the body axis. Then, the dynamic model of body rates is expressed as follows:

$$\dot{\Omega}(t) = J^{-1}\left(\tau(t) - (\Omega(t)\times J\Omega(t))\right). \tag{2}$$

We apply a preliminary feedback

$$\tau(t) = (\Omega(t)\times J\Omega(t)) + Ju(t),$$

where $u(t)\in\mathbb{R}^3$. This simplifies (2) to $\dot{\Omega} = u$, and the rigid body dynamics simplify to

$$\begin{aligned} \dot{R}(t) &= R(t)\widehat{\Omega}(t), \\ \dot{\Omega}(t) &= u(t). \end{aligned} \tag{3}$$

The above system (3) represents a fully-actuated rigid body capable of rotation about a point without translation such that the states of the system $(R,\Omega)\in\mathsf{SO}(3)\times\mathbb{R}^3$. To vectorize an $n\times n$ matrix, we define a linear bijective map:

$$\lozenge : \mathbb{R}^{n\times n} \to \mathbb{R}^{n^2}$$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \mapsto \begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \\ a_{1n} \\ \vdots \\ a_{nn} \end{bmatrix}.$$

Although $\lozenge$ is bijective on $\mathbb{R}^{3\times 3}$, it is not invertible when its domain is restricted to $\mathsf{SO}(3)$. The redefinition of $\lozenge$ on a restricted domain solves this issue, i.e.,

$$\lozenge|_{\mathsf{SO}(3)} : \mathsf{SO}(3) \to \mathrm{Im}(\lozenge_{\mathsf{SO}(3)}).$$

It should be noted that $\lozenge|_{\mathsf{SO}(3)}$ is a diffeomorphism [11]. Let r_{ij} be the element of a matrix $R\in\mathsf{SO}(3)$. Next, we apply

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[1] Given $\omega\in\mathbb{R}^3$, we express $\widehat{\omega}\in\mathfrak{so}(3)$ or equivalently $(\omega)^\wedge\in\mathfrak{so}(3)$.

$\Diamond|_{\mathsf{SO}(3)}$ on both sides of (1), i.e., $\Diamond|_{\mathsf{SO}(3)}(\dot{R}) = \Diamond|_{\mathsf{SO}(3)}(R\widehat{\Omega})$. This allows us to express (3) as

$$
\begin{aligned}
\dot{r}_{k1} &= r r_{k2} - q r_{k3}, & \dot{p} &= u_1 \\
\dot{r}_{k2} &= p r_{k3} - r r_{k1}, & \dot{q} &= u_2 \\
\dot{r}_{k3} &= q r_{k1} - p r_{k2}, & \dot{r} &= u_3
\end{aligned}
\tag{4}
$$

$\forall k \in \{1, 2, 3\}$. It should be noted that the above system of equation is not the minimal representation but leads to a considerably simpler set of equations as compared to the use of five or less than five parameters [2]. We define state of the system as $x := \mathrm{col}(\Diamond|_{\mathsf{SO}(3)}(R), \Omega) \in \mathbb{R}^{12}$. Let

$$
f(x) := \begin{bmatrix} r r_{k2} - q r_{k3} \\ p r_{k3} - r r_{k1} \\ q r_{k1} - p r_{k2} \\ 0_{3,1} \end{bmatrix}, g(x) := \begin{bmatrix} 0_{9,3} \\ I_3 \end{bmatrix}, \forall k \in \{1, 2, 3\}
$$

then we can write (4) in the control-affine form as follows:

$$
\dot{x} = f(x) + g(x)u. \tag{5}
$$

Next we formulate the tracking problem of rigid body system.

III. PROBLEM FORMULATION

A control stabilization problem is stated as follows: given a system from an initial position, design a controller that makes the system converge to a static desired orientation. In contrast, in a tracking problem the desired orientation is a time varying function. Therefore, the stabilization problem can be considered as a special case of the tracking problem when t is constant. In this work, we consider the rigid body stabilization problem, and without loss of generality, we assume that the desired orientation is $R = I_3$, or the upright position of the rigid body. Let $\overline{x} := (I_3, 0_{3,1}) = (\Diamond|_{\mathsf{SO}(3)}(I_3), 0_{3,1}) \in \mathbb{R}^{12}$ is the desired state. Before formally presenting the problem statement, we define the domain of the state space as

$$
\mathbb{D} := \mathbb{D}_{\mathsf{SO}(3)} \times \mathbb{R}^3,
$$

where $\mathbb{D}_{\mathsf{SO}(3)} := \mathsf{SO}(3) / \{\mathrm{diag}(-1, -1, 1), \mathrm{diag}(-1, 1, -1), \mathrm{diag}(1, -1, -1)\}$. We have removed three points, which represent exactly upside down position of the rigid body from $\mathsf{SO}(3)$.

Problem Statement: Given a fully-actuated rigid body (3), and an open set of initial conditions in the neighborhood of $\overline{x}$, design a continuous feedback control law $u : \mathbb{D} \to u(\mathbb{D})$ such that the system asymptotically approaches the desired point $\overline{x}$, i.e., $\|x(t) - \overline{x}\| \to 0$, as $t \to \infty$.

We call a controller a *global converging controller* if it makes the system converge from an arbitrary point of the state space to the desired point. Similarly, we call a controller an *almost-global converging controller* if it makes the system converge to the desired point from every point of the state space except a "small" set of Lebesgue measure zero. Moreover, we call a controller a *local converging controller* if it only works in a small neighborhood of the desired point. Our approach is flexible because it allows to fit both local and almost-global converging controllers in one setting. If the neighborhood of $\overline{x}$ consists of the entire $\mathbb{D}$, we get almost-global convergence, else we achieve local convergence.

IV. CONTROLLER DESIGN

Given the rigid body dynamics (3), our control design technique consists of two steps:

1) apply $\Diamond|_{\mathsf{SO}(3)}$ to (3) and express the system in control-affine form (4) and
2) pick a function of state variables and perform partial state feedback linearization.

It should be noted that the second step is non-trivial and it entails selecting three functions or one vector-valued function of state variables. Since this function will be used to perform partial feedback linearization, we call such a function a feedback linearization function. Next, we present two novel definitions of feedback linearization functions: an almost-global feedback linearization function denoted by $\mathcal{L}_f^A$, and a local feedback linearization function denoted by $\mathcal{L}_f^L$.

Definition IV.1. *An analytic function* $\alpha : \mathbb{D}_{\mathsf{SO}(3)} \to \mathbb{R}^3$ *is an almost-global feedback linearizing function* $\mathcal{L}_f^A$ *if*

$$
L_f \alpha = c\Omega, \tag{6}
$$

for all $c \in \mathbb{R}/\{0\}$ *everywhere on* $\mathbb{D}$ *and* $\alpha^{-1}(0_3) = I_3$.

The following result is the direct consequence of Definition IV.1:

Corollary IV.2. *For each* $\alpha \in \mathcal{L}_f^A$ *the following statements hold:*

1) $L_f^2 \alpha = 0_3$,
2) $\ker(L_g L_f \alpha) = 0_3$,

everywhere on $\mathbb{D}$.

Proof. We prove the result by construction. First compute the Jacobian of $L_f \alpha$

$$
\begin{aligned}
&J_x L_f \alpha : \mathbb{R}^3 \to \mathbb{R}^{3 \times 12} \\
&\qquad L_f \alpha \mapsto \frac{\partial}{\partial x} L_f \alpha = \left[\frac{\partial}{\partial (\Diamond|_{\mathsf{SO}(3)}(R))} L_f \alpha \quad \frac{\partial}{\partial \Omega} L_f \alpha \right].
\end{aligned}
$$

Since for every $\alpha \in \mathcal{L}_f^A$, $L_f \alpha$ is strictly a function of Ω, therefore, we can write

$$
\left[\frac{\partial}{\partial (\Diamond|_{\mathsf{SO}(3)}(R))} L_f \alpha \right] = 0_{3,9}
$$

and

$$
\left[\frac{\partial}{\partial \Omega} L_f \alpha \right] = c I_3.
$$

Using the definition of Lie derivative

$$
\begin{aligned}
L_f^2 \alpha &= [J_x L_f \alpha]\, f \\
&= [0_{3,9} \ \ I_3]\, f \\
&= 0_3.
\end{aligned}
\tag{7}
$$

This proves the first condition of Corollary IV.2. To prove the second condition, we compute

$$\begin{aligned} L_g L_f \alpha &= [J_x L_f \alpha]\,[g] \\ &= [0_{3,9} \;\; cI_3]\,[g] \\ &= cI_3. \end{aligned} \tag{8}$$

Since $c \neq 0$, this implies that $\ker(L_g L_f \alpha) = 0_3$. This completes the proof. ☐

Remark IV.3. *Definition IV.1 imposes strict conditions on $\alpha \in \mathcal{L}_f^A$. The function needs to be analytic and satisfy (6). This strict condition reduces the choices of $\mathcal{L}_f^A$ functions.*

We present our second novel definition in the sequel and call this a local feedback linearization function.

Definition IV.4. *A function $\alpha\colon \mathbb{D}_{\mathsf{SO}(3)} \to \mathbb{R}^3$ is a $\mathcal{L}_f^L$ function if it satisfies the following conditions in an open neighborhood of $\overline{x}$:*

1) *α is at least C^2,*
2) *$\ker(L_g L_f \alpha) = 0_3$,*
3) *$\alpha^{-1}(0_3) = I_3$.*

We define a set on the state space $\mathbb{R}^{12}$ containing $\overline{x}$ as

$$\mathcal{N}_{\overline{x}} := \{x \in \mathbb{R}^{12} : \ker(L_g L_f \alpha) = 0_3\} \tag{9}$$

and call this $\overline{x}$-neighborhood.

Remark IV.5. *Definition IV.4 allows less strict conditions on the continuity of α. Moreover, this definition entails conditions that need to be satisfied only in the $\overline{x}$-neighborhood. These relaxed conditions allow more choices of $\mathcal{L}_f^L$ compared to $\mathcal{L}_f^A$.*

These function definitions yield the following result:

Lemma IV.6. *Given a function $\alpha\colon \mathbb{D}_{\mathsf{SO}(3)} \to \mathbb{R}^3$ belonging to $\mathcal{L}_f^A$, the rigid body system (3) has a vector relative degree of $\{2,2,2\}$ at each point on $\mathbb{D}$. Moreover, if $\alpha \in \mathcal{L}_f^L$ and given $\mathcal{N}_{\overline{x}}$ defined by (9), the system (3) has a vector relative degree of $\{2,2,2\}$ at each point on $\mathcal{N}_{\overline{x}}$.*

Proof. Let $x^\star \in \mathcal{N}_{\overline{x}}$ be arbitrary. By the definition of vector relative degree, we must show that

1) $L_g \alpha(x^\star) = 0_3$, and
2) $L_g L_f \alpha(x^\star)$ is invertible on $\mathcal{N}_{\overline{x}}$.

For α belonging to either $\alpha \in \mathcal{L}_f^A$ or $\alpha \in \mathcal{L}_f^L$, the domain is $\mathbb{D}_{\mathsf{SO}(3)}$, which consists of only $R \in \mathsf{SO}(3)$. Therefore, the flow of α in the direction of g is zero, i.e., $L_g \alpha(x^\star) = 0_3$. By Corollary IV.2, $\ker(L_g L_f \alpha) = 0_3$ everywhere on $\mathbb{D}$, which implies that $L_g L_f \alpha$ is non-singular everywhere on $\mathbb{D}$. Moreover, by Definition IV.4 $\ker(L_g L_f \alpha) = 0_3$ everywhere on the $\overline{x}$-neighbourhood, and is therefore non-singular everywhere on $\mathcal{N}_{\overline{x}}$. Therefore, the system has a vector relative degree of {2,2,2} at $x^\star$ everywhere on $\mathcal{N}_{\overline{x}}$ when $\alpha \in \mathcal{L}_f^L$. Moreover, the system has a vector relative degree of {2,2,2} at $x^\star$ everywhere on $\mathbb{D}$ when $\alpha \in \mathcal{L}_f^A$. ☐

Remark IV.7. *Under the light of Lemma IV.6, every $\alpha \in \mathcal{L}_f^A$ ensures invertibility of $L_g L_f \alpha$ everywhere on $\mathbb{D}$ and every $\alpha \in \mathcal{L}_f^L$ ensures invertibility of $L_g L_f \alpha$ everywhere in the $\overline{x}$-neighborhood. The invertibility of $L_g L_f \alpha$ is crucial for the controller design.*

Next, we present two detailed examples of α functions. The first example is an almost-global feedback linearizing function $\alpha \in \mathcal{L}_f^A$ and the second example is a local feedback linearizing function $\mathcal{L}_f^L$.

Example IV.8. *In this example, we select a function*

$$\begin{aligned} \alpha\colon \mathbb{D}_{\mathsf{SO}(3)} &\to \mathbb{R}^3 \\ R &\mapsto \left(\operatorname{Log}(R)\right)^\vee, \end{aligned} \tag{10}$$

and argue that this is an almost-global feedback linearizing function, i.e., $\alpha \in \mathcal{L}_f^A$. To this end, we need to check if it satisfies Definition IV.1. It is easy to check that the function is analytic as matrix log is analytic on $\mathbb{D}$, and $(\operatorname{Log}(I_3))^\vee = 0_3$. Next, we compute the derivative of (10). Direct calculation gives

$$\dot{\alpha} = L_f \alpha + (L_g \alpha) u.$$

By Definition IV.1, the domain of α is $\mathbb{D}_{\mathsf{SO}(3)}$ so $L_g \alpha = 0_{3,3}$. In other words, the pre-image of α does not include Ω, so the flow of α along the vector field g must be zero. Therefore, $\dot{\alpha} = L_f \alpha$. By Proposition VII.1 $\frac{d}{dt}(\operatorname{Log}(R)) = \widehat{\Omega}$, which implies $L_f \alpha = \Omega$. Therefore, $R \mapsto (\operatorname{Log}(R))^\vee$ satisfies Definition IV.1 and is a valid almost-global feedback linearization function. Next, we investigate the vector relative degree by direct computation. The second derivative of α gives

$$\begin{aligned} \ddot{\alpha} &= L_f^2 \alpha + (L_g L_f \alpha) u \\ &= \begin{bmatrix} 0_{3,9} & I_3 \end{bmatrix} \begin{bmatrix} r r_{12} - q r_{13} \\ r r_{22} - q r_{23} \\ r r_{32} - q r_{33} \\ p r_{13} - r r_{11} \\ p r_{23} - r r_{21} \\ p r_{33} - r r_{31} \\ q r_{11} - p r_{12} \\ q r_{21} - p r_{22} \\ q r_{31} - p r_{32} \\ 0_{3,1} \end{bmatrix} + \begin{bmatrix} 0_{3,9} & I_3 \end{bmatrix} \begin{bmatrix} 0_{9,3} \\ I_3 \end{bmatrix} u \\ &= 0_{3,1} + I_3 u. \end{aligned}$$

This shows that $L_g L_f \alpha = I_3$ and is invertible at every point on $\mathbb{D}$, as suggested by Lemma IV.6. Therefore, the system possesses a vector relative degree of $\{2,2,2\}$ everywhere on $\mathbb{D}$ with this particular choice of function.

Example IV.9. *In this example, we select another feedback linearizing function*

$$\begin{aligned} \alpha\colon \mathbb{D}_{\mathsf{SO}(3)} &\to \mathbb{R}^3 \\ R &\mapsto \begin{bmatrix} r_{23} - r_{32} \\ r_{31} - r_{13} \\ r_{12} - r_{21} \end{bmatrix}, \end{aligned} \tag{11}$$

and will show that it is a local feedback linearizing function, i.e., $\alpha \in \mathcal{L}_f^L$ and satisfies Definition IV.4. It is easy to check that α given by (11) is C^2 at $\overline{x}$, and by continuity, it is C^2 in the neighborhood of $\overline{x}$. To check the second condition of Definition IV.4, we compute the standard Lie derivatives.

$$\begin{aligned}\dot{\alpha} &= L_f\alpha + (L_g\alpha)u \\ &= \begin{bmatrix} qr_{21} - pr_{33} - pr_{22} + rr_{31} \\ pr_{12} - qr_{11} - qr_{33} + rr_{32} \\ pr_{13} + qr_{23} - rr_{11} - rr_{22} \end{bmatrix} + 0_{3,3}u,\end{aligned}$$

where $L_g\alpha = 0_{3,3}$ is a direct consequence of the definition. To show the second condition of Definition IV.4, we take the second derivative of α.

$$\begin{aligned}\ddot{\alpha} &= L_f^2\alpha + (L_gL_f\alpha)u \\ &= \begin{bmatrix} p(pr_{32} - qr_{31}) - p(pr_{23} - rr_{21}) \\ p(pr_{13} - rr_{11}) + q(pr_{32} - qr_{31}) \\ r(qr_{13} - rr_{12}) - q(pr_{22} - qr_{21}) \end{bmatrix} \\ &+ \begin{bmatrix} -q(qr_{23} - rr_{22}) - r(qr_{33} - rr_{32}) \\ +q(qr_{13} - rr_{12}) + r(pr_{33} - rr_{31}) \\ -r(pr_{23} - rr_{21}) - p(pr_{12} - qr_{11}) \end{bmatrix} \\ &+ \begin{bmatrix} -r_{22} - r_{33} & r_{21} & r_{31} \\ r_{12} & -r_{11} - r_{33} & r_{32} \\ r_{13} & r_{23} & -r_{11} - r_{22} \end{bmatrix} u,\end{aligned}$$

where the first two terms in the above expression represent $L^2 f\alpha$. It is easy to check that $L_gL_f\alpha|_{\overline{x}=(I,0_{3,1})} \in \mathrm{GL}(3,\mathbb{R})$ as $L_gL_f\alpha|_{\overline{x}=(I_3,0_{3,1})} = \mathrm{diag}(-2,-2,-2)$. Hence by uniform continuity of α, $L_gL_f\alpha$ is invertible in the neighborhood of $\overline{x}$. Therefore, $\ker(L_gL_f\alpha) = 0$ everywhere on $\mathcal{N}_{\overline{x}}$, and this satisfies the second condition of Definition IV.4. It can be seen that the third condition is also true as $\alpha^{-1}(0_3) = I_3$. By Lemma IV.6 and with the choice of α in (11), the system has a vector relative degree of $\{2,2,2\}$ in the neighborhood of $\overline{x}$.

Remark IV.10. *The second example emphasizes the fact that the region where system achieves a well-defined vector relative degree depends on the choice of $\alpha \in \mathcal{L}_f^L$.*

The system achieves a vector relative degree of $\{2,2,2\}$ with an appropriate choice of α. The resulting dimension of the internal dynamics of the system becomes 6, which implies that the coordinate transformation can be completed by choosing six additional functions. A possible choice for these internal states is as follows:

$$\mu := \mathrm{col}(r_{11}, r_{21}, r_{31}, r_{12}, r_{22}, r_{32}) \tag{12}$$

Corollary IV.11. *Let $\alpha \in \mathcal{L}_f^L$ and $x^\star \in \mathcal{N}_{\overline{x}}$. There exists a neighbourhood $U \subset \mathcal{N}_{\overline{x}}$, that contains $x^\star$ and $\overline{x}$, such that the mapping $T : U \subset \mathcal{N}_{\overline{x}} \to T(U) \subset \mathbb{R}^{12}$, defined by*

$$\begin{bmatrix} \xi \\ \eta \\ \mu_i \end{bmatrix} = T(x) = \begin{bmatrix} \alpha(x) \\ L_f\alpha(x) \\ \mu(x) \end{bmatrix}, \tag{13}$$

for $i \in \{1,\cdots,6\}$ is a local diffeomorphism. Moreover, when $\alpha \in \mathcal{L}_f^A$ and the point $x^\star \in \mathbb{D}_{\mathsf{SO}(3)}$, then $T : U \subset D_{\mathsf{SO}(3)} \to T(U) \subset \mathbb{R}^{12}$ is a diffeomorphism.

The proof has been omitted due to space limitations, however, cf. [11, Corollary 6] for a similar result about underactuated systems. Let $\Xi := (\xi, \eta, \mu)$. By applying the diffeomorphic map T from Corollary IV.11, then, in a neighbourhood of $x^\star$, the system can be expressed as

$$\begin{aligned}\dot{\xi} &= \eta \\ \dot{\eta} &= L_f^2\alpha + L_gL_f\alpha u|_{x=T^{-1}(\Xi)} \\ \dot{\mu}_j &= b_j(\Xi)|_{x=T^{-1}(\Xi)}\end{aligned} \tag{14}$$

for $j \in \{1,\cdots,6\}$, where b_j are smooth functions. It should be noted that for $\alpha \in \mathcal{L}_f^A$, this coordinate transformation is defined everywhere on $\mathbb{D}_{\mathsf{SO}(3)}$. In other words, for $\alpha \in \mathcal{L}_f^A$ the coordinate transformation is defined everywhere on $\mathbb{D}$ except on a small set of Lebesgue measure zero, and we say (14) is almost-globally differentially equivalent to (4). Otherwise, if $\alpha \in \mathcal{L}_f^L$ the coordinate transformation is valid everywhere on $\mathcal{N}_{\overline{x}}$, and we say (14) is locally differentially equivalent to (4). From (14), we can write

$$u := (L_gL_f\alpha)^{-1}\left(-L_f^2\alpha + v^\eta\right), \tag{15}$$

where $v^\eta := \mathrm{col}(v^{\eta_1}, v^{\eta_2}, v^{\eta_3}) \in \mathbb{R}^3$ are auxiliary control inputs. By Lemma IV.6, for $\alpha \in \mathcal{L}_f^A$ the map (15) is well-defined in a neighborhood of every $x^\star \in \mathbb{D}_{\mathsf{SO}(3)}$ and for $\alpha \in \mathcal{L}_f^L$ the feedback transformation (15) is well-defined in a neighborhood of every $x^\star \in \mathcal{N}_{\overline{x}}$. Thus depending on α the closed-loop system is reduced, either locally or almost-globally, to a chain of integrators and uncontrolled nonlinear internal dynamics

$$\dot{\xi}_i = \eta_i, \quad \dot{\eta}_i = v^{\eta_i}, \quad \dot{\mu}_j = b_j(\Xi), \tag{16}$$

where v^{η_i} for $i \in \{1,2,3\}$ are auxiliary control inputs. It is straight forward to design a linear controller for the above mentioned dynamics expressed in partial-linear form

$$v^{\eta_i} = -k_{1i}\xi_i - k_{2i}\eta_i, \tag{17}$$

for some positive constants k_{1i}, k_{2i}, $i \in \{1,2,3\}$ to stabilize the origin of the ξ, η system.

V. INTERNAL DYNAMICS

The internal dynamics of the system are expressed by the μ-subsystem as follows:

$$\begin{aligned}\dot{\mu}_i(\Xi) &= rr_{i2} - qr_{i3}|_{x=T^{-1}(\Xi)} \\ \dot{\mu}_{i+3}(\Xi) &= pr_{i3} - rr_{i1}|_{x=T^{-1}(\Xi)}\end{aligned} \tag{18}$$

Next results shows stability of the internal dynamics.

Lemma V.1. *The internal states and dynamics in (18) are bounded given the control inputs are bounded.*

Proof is omitted due to space limitations. In summary, the linear controller (17) makes all ξ and η states exponentially converge to zero. This means that α and $\dot{\alpha}$ also converge to zero. By Definition IV.1 and Definition IV.4, the system converges to I_3 with all body rates converge to zero. Moreover, Definition IV.1 ensures almost-global convergence to the desired point, while Definition IV.4 ensures local convergence to the desired point. Therefore, we have solved the stabilization problem of the rotating rigid body.

VI. SIMULATION RESULTS

Simulation results for of almost-global controllers are presented in this section, while the local controller results are omitted because of space limitations.

A. *Almost-global feedback linearizing controller*

Now we present simulation results of almost-global controller introduced in Example IV.8. We initialize the system with an "almost" upside down position. One way to initialize the system at this position is by giving a roll angle $\phi = 179°$. It is assumed that at the initial time all body rates are zero, i.e., $\Omega = \mathrm{col}(0, 0, 0)$. Figure 1 indicates that the system recovers from a large initial error and then converges to zero. Figure 2 shows that the transformed states also converge to zero, and Figure 3 shows that all body rates converge to zero. In other words, the system has converged to the desired point $\overline{x}$.

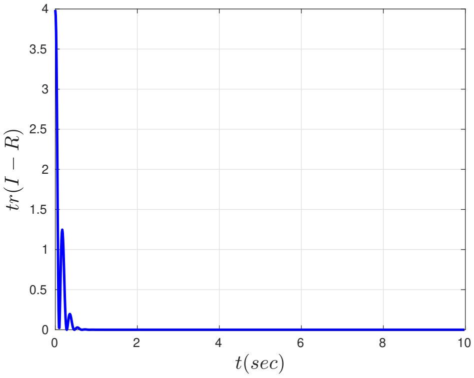

Fig. 1: Stabilization error $\mathrm{tr}(I - R)$

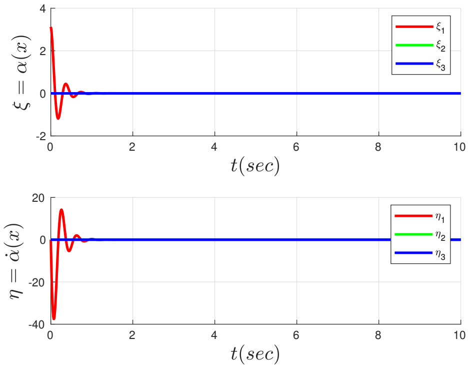

Fig. 2: Transformed states $(\xi, \eta) \to 0$ asymptotically

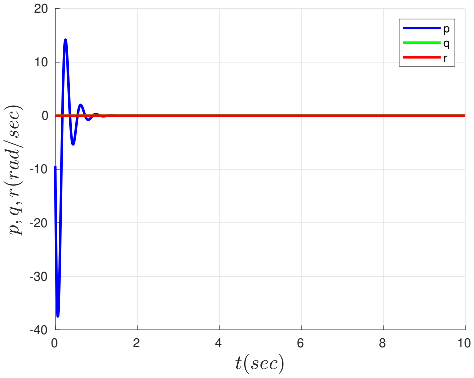

Fig. 3: Body rates $\Omega \to 0$ asymptotically

VII. APPENDIX

Proposition VII.1. *Let* $R(t) \in \mathsf{SO}(3)$ *satisfying*

$$\dot{R}(t) = R(t)\widehat{\Omega}(t),$$

then $\frac{\mathrm{d}}{\mathrm{d}\,t}\operatorname{Log}(R(t)) = \left(R^{\top}\right)\dot{R} = \widehat{\Omega}(t)$

Proof. For proof, the readers are referred to [12]. ☐

Lemma VII.2. *The system* (19)

$$J\dot{\Omega} = \tau - (\Omega \times J\Omega) \tag{19}$$

is input-to-state stable.

Proof. For proof, the readers are referred to [4, Lemma VI.3]. ☐

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