Path Following for a Quadrotor using Dynamic Extension and Transverse Feedback Linearization

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***Abstract*— This work presents a path following controller for a quadrotor vehicle. A smooth, dynamic, feedback controller is designed that allows the quadrotor to follow both closed and non-closed embedded curves while maintaining a desired speed, a desired acceleration or while stabilizing desired points along the curves. The nonlinear dynamic model of the quadrotor is transformed into a linear system via a coordinate and feedback linearization transformation. Once transformed, a path following controller is designed that guarantees invariance of the path while enforcing the desired motion along the path.**

I. Introduction

Unmanned Aerial Vehicles (UAVs) have become increasingly popular in both academia and industry because of their wide range of applications in both tactical and commercial sectors. In particular, small rotary-wing UAVs e.g., quadrotor helicopters, have attracted researchers in the last decade because of their compact size, low cost and the ability to operate safely in indoor environments or in the presence of humans. For these types of applications, however, precise control performance is required to make the quadrotor hover at a specific point or follow desired paths or trajectories with minimal deviation.

In [1] the authors show that one advantage of designing a path following controller is that it guarantees invariance of the path, ensuring that once the vehicle has converged to the path, safe operation along the path can be guaranteed.

A wide range of nonlinear techniques have also been proposed to address the envelope restrictions inherent in linear controller designs. Backstepping [2], sliding mode [3], [4] and feedback linearization [4]–[6] methods have been presented, with varying choices on the controller structure being explored to deal with the underactuated nature of the quadrotor platform. Some flight test comparisons have been made between these designs, for example between nested saturation, backstepping and sliding mode controllers [7] and between feedback linearization and adaptive sliding mode [4]. In both cases, the results suggest that incorporating robust control elements are of significant benefit for real world performance.

An alternative approach to solving the precise motion problem relies on using the concept of path following. It is shown in [8] that by treating path following as a set stabilization problem, the desired path can be made invariant. In this paper, we solve the path following problem of a quadrotor using the concept of transverse feedback linearization [1], [9], which has successfully been applied to a planar vertical take-off and landing aircraft model [10] and to car-like robots [1]. In this work, the quadrotor model is converted to a fully linear form via dynamic extension and a nonlinear coordinate transformation, which implies that we have identified a set of differentially flat outputs [11]. It is important to emphasize that the outputs selected in this paper have intuitive geometric meaning and are natural choices for solving the path following problem.

This work is similar to but independent from recent results presented by A. Roza and M. Maggiore [12]. The following three extensions are presented in this work. First, the controller allows the quadrotor to move along the path is any desired manner, including stopping along the curve as well as changing the direction of traversal along the path. Second, we fully linearize the system and the dimension of the zero dynamics is therefore zero. Finally, both closed and non-closed curves can be used for the path definition, resulting in a more general class of paths that can be followed.

1) Notation: The point-to-set distance from a $x \in \mathbb{R}^2$ to a set $\Gamma \subset \mathbb{R}^n$ is denoted by $\|x\|_\gamma$, .i.e.,

$$\|x\|_\gamma := \inf_{p \in \Gamma} \|x - p\|$$

where $\|\cdot\|$ is the Euclidean norm. The trigonometric functions are abbreviated as $s_i := \sin(x_i)$, $c_i := \cos(x_i)$, $\tan_{11} := \tan(x_{11})$ and $\sec_{11} := \sec(x_{11})$. Let $s \circ h : A \to C$ represent the composition of maps $s : B \to C$ and $h : A \to B$. Let $\mathrm{col}(x_1, \ldots, x_k) := \begin{bmatrix} x_1 & \cdots & x_k \end{bmatrix}^\top$ where $^\top$ denotes transpose. Given a C^1 map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and a point $p \in \mathbb{R}^n$, we denote $df_x := \frac{\partial f}{\partial x}(p)$. If f, $g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}$ are smooth, we use the following standard notation for iterated Lie derivatives $L_g L_f \lambda := L_g(L_f \lambda)$, $L_g^0 \lambda := \lambda$, $L_g^k \lambda := L_g(L_g^{k-1}\lambda)$.

II. Dynamic model of the quadrotor

A quadrotor is a mechatronic system with four propellers in a cross configuration, as depicted in Figure 1. It is a nonlinear system consisting of four inputs (the thrust provided by each propeller) and six degrees-of-freedom (the motion in three translational and three rotational DOFs), and is therefore an example of an underactuated system. The dynamic model of a quadrotor has been discussed in great detail and in this paper, a brief overview is presented. The readers are referred to [13]–[15] for further details. Let

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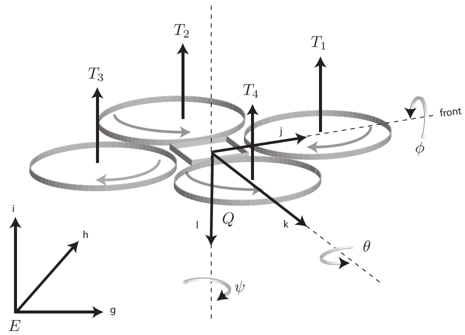

Fig. 1. Quadrotor schematics showing body frame Q and inertial frame E along with forces, moments, and Euler angles.

X, Y, Z represent the position of the quadrotor in the inertial frame E and roll, pitch and yaw Euler angles (ϕ, θ and ψ respectively) represent the orientation of the quadrotor relative to the inertial frame as shown in Figure 1. The body angular rates are represented by p, q and r in the body frame Q. We define the state vector as $x = \mathrm{col}(x_1, \ldots, x_{12}) := \mathrm{col}(X, \dot{X}, Y, \dot{Y}, Z, \dot{Z}, p, q, r, \phi, \theta, \psi) \in \mathbb{R}^{12}$ and the control input vector as $\bar{u} := \mathrm{col}(u_c, u_\phi, u_\theta, u_\psi) \in \mathbb{R}^4$. The dynamic model of a quadrotor can be expressed compactly in control affine form $\dot{x} = f(x) + \sum_{i=1}^{4} g_i(x)\overline{u}_i$ as

$$\dot{x} = \begin{bmatrix} x_2 \\ 0 \\ x_4 \\ 0 \\ x_6 \\ -g \\ x_8 x_9 I_1 \\ x_7 x_9 I_2 \\ x_7 x_8 I_3 \\ x_7 + \tan_{11} R_4 \\ x_8 c_{10} - x_9 s_{10} \\ \sec_{11} R_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ R_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ R_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ R_3 & 0 & 0 & 0 \\ 0 & C_x & 0 & 0 \\ 0 & 0 & C_y & 0 \\ 0 & 0 & 0 & C_z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_c \\ u_\phi \\ u_\theta \\ u_\psi \end{bmatrix} \tag{1}$$

In (1), we set $I_1 := \left(\frac{I_y - I_z}{I_x}\right)$, $I_2 := \left(\frac{I_z - I_x}{I_y}\right)$ and $I_3 := \left(\frac{I_x - I_y}{I_z}\right)$ where I_x, I_y and I_z represent the body moment of inertia terms. Let $C_x := L/I_x$, $C_y := L/I_y$, $C_z := 1/I_z$ where $L > 0$ denotes the horizontal distance from the center of mass of the quadrotor to the motors. The terms R_i denote the smooth functions $R_i : \mathbb{R}^{12} \longrightarrow \mathbb{R}$, $i \in \{1, \ldots, 4\}$, defined as

$$\begin{aligned} R_1 &:= \frac{1}{m}\left(-c_{10}s_{11}s_{12} + s_{10}c_{12}\right) \\ R_2 &:= \frac{1}{m}\left(-c_{10}s_{11}c_{12} - s_{10}s_{12}\right) \\ R_3 &:= \frac{1}{m}\left(c_{10}c_{11}\right) \\ R_4 &:= x_8 s_{10} + x_9 c_{10}, \end{aligned}$$

where m is the mass of the quadrotor and g is the acceleration due to gravity. Let T_i, $i \in \{1, \ldots, 4\}$, represent the thrust of the i^{th} rotor, then the control input $\bar{u}$ is related to the thrust forces via

$$\begin{bmatrix} u_c \\ u_\phi \\ u_\theta \\ u_\psi \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix}.$$

We take the position of the center of mass of quadrotor in the intertial frame as the output of (1)

$$y = h(x) = \begin{bmatrix} x_1 & x_3 & x_5 \end{bmatrix}^\top. \tag{2}$$

III. Problem Statement

Informally, path following entails making the output of the system approach and move along a given path with no pre-specified timing law associated with the motion along the path.

Assumption 1: The desired path is a smooth, regular, parameterized curve

$$\begin{aligned} \sigma &: \mathbb{D} \to \mathbb{R}^3 \\ \lambda &\mapsto \begin{bmatrix} \sigma_1(\lambda) \\ \sigma_2(\lambda) \\ \sigma_3(\lambda) \end{bmatrix}. \end{aligned} \tag{3}$$

The set $\sigma(\mathbb{D})$ is assumed to be an embedded submanifold of $\mathbb{R}^3$. We also assume that there exists a smooth map $s : \mathbb{R}^3 \to \mathbb{R}^2$ so that $\sigma(\mathbb{D}) = s^{-1}(0)$ with 0 a regular value of s.

By Assumption 1, σ is a regular curve so that without loss of generality, we henceforth assume that σ is parameterized by its arc length, i.e., $\|\sigma'\| \equiv 1$. We use the symbol $\mathbb{D}$ to denote the domain of σ. For non-closed curves $\mathbb{D} = \mathbb{R}$. For closed curves with finite length L, this means that $\mathbb{D} = \mathbb{R} \bmod L$ and σ is L-periodic, i.e., for any $\lambda \in \mathbb{D}$, $\sigma(\lambda + L) = \sigma(L)$.

If $\sigma(\mathbb{D})$ is an embedded submanifold of $\mathbb{R}^3$, then it is always possible to locally represent the curve as the zero level set of a function. Assumption 1 ensures that the *entire* path can be represented as the zero level set of a smooth function. Let $\gamma := s^{-1}(0)$, then, in the case of the quadrotor system (1), the path is represented in the output space as

$$\gamma := \left\{ y \in \mathbb{R}^3 : s_1(y) = s_2(y) = 0 \right\}.$$

Similar to the previous work [1], the lift of the path γ to $\mathbb{R}^{12}$ is defined as

$$\Gamma := \left\{ x \in \mathbb{R}^{12} : s_1(h(x)) = s_2(h(x)) = 0 \right\}.$$

The control objective is to make the output y of the system (1) asymptotically converge and then follow the path. Making $y \to \gamma$ is equivalent to making $x \to \Gamma$. However, we will see that in general Γ can not be made invariant and hence we will try to stabilize a subset of Γ.

Given a path that satisfies Assumption 1, we seek a smooth dynamic feedback law

$$\begin{aligned} \dot{\zeta} &= \mathcal{A}(x,\zeta) + \mathcal{B}(x,\zeta)u \\ \overline{u} &= \mathcal{C}(x,\zeta) + \mathcal{D}(x,\zeta)u, \end{aligned} \tag{4}$$

with[1] $\zeta \in \mathbb{R}^{\tilde{q}}$, $u = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4$ and an open subset of initial conditions $U \times V \in \mathbb{R}^{12} \times \mathbb{R}^{\tilde{q}}$ with $\gamma \subset h(U)$, such that for any initial condition $(x(0), \zeta(0)) \in U \times V$ the corresponding solution $x(t)$ for the closed-loop system is defined for all $t \geq 0$ and

PF1 The quadrotor approaches the path, $\|h(x(t))\|_\gamma \to 0$ as $t \to \infty$.

PF2 The level set $s(y)$ is output invariant, i.e., if the quadrotor is initialized on the path with output velocity tangent to the path, it remains on the path for all $t \geq 0$.

PF3 On the path, the system meets additional application specific requirements such as
- Stabilizing a desired point along the path
- Tracking a desired speed and/or acceleration profile along the curve.
- Tracking a desired yaw angle value or reference profile along the curve.

IV. Dynamic Extension

To solve the path following problem we seek to find the largest controlled invariant subset of Γ. As discussed in [1], [9], [16], the largest controlled invariant submanifold $\Gamma^\star$ contained in Γ is called the path following manifold. It consists of all those trajectories of the system whose associated output signal can be made to remain on the desired path by a suitable choice of the control input. The path following manifold plays a key role in designing path following controllers because if $\Gamma^\star$ can be made attractive then **PF1** and **PF2** are achieved. In order to find $\Gamma^\star$ we first define

$$\alpha := s \circ h(x) = \begin{bmatrix} s_1 \circ h(x) \\ s_2 \circ h(x) \end{bmatrix} = \begin{bmatrix} \alpha_1(x) \\ \alpha_2(x) \end{bmatrix}. \tag{5}$$

With this definition we have that $\Gamma = \alpha^{-1}(0)$ and as a result, we can apply the zero dynamics algorithm to the function α to obtain a local characterization of $\Gamma^\star$.

Given that the quadrotor has four inputs, it is natural to augment the function (5) with two additional "virtual outputs" and then check if this resulting virtual output has a well-defined vector relative degree. To this end let $\pi_1(x_1, x_3, x_5) : \mathbb{R}^3 \to \mathbb{R}$ be a map defined in the output space of the quadrotor. We restrict our choice of π_1 to have the form $\varpi \circ h(x)$ because, as will be shown in subsequent development, it plays the role of defining "where" on the given path the quadrotor is. This choice makes the output π_1 critical for meeting the specifications in **PF3** that involve tracking a desired velocity or acceleration profile on the curve or stabilizing a particular point on the curve. Finally, we choose $\pi_2(x_1, x_3, x_5, x_{12}) : \mathbb{R}^4 \to \mathbb{R}$. This choice is motivated by the fact that, according to **PF3** , we would like the yaw angle x_{12} to be virtually constrained by the position of the quadrotor along the path, which is completely specified by the values of (x_1, x_3, x_5).

Assumption 2: The function $\pi_2(x_1, x_3, x_5, x_{12}) : \mathbb{R}^4 \to \mathbb{R}$ satisfies

$$\frac{\partial \pi_2}{\partial x_{12}} \neq 0$$

for every $x_{12} \in [0, 2\pi)$ and every $(x_1, x_3, x_3) \in \mathbb{R}^3$ such that $h(x) \in \gamma$.

Assumption 2 ensures that, at each point along the path, we can apply the implicit function theorem on $\overline{y}_4 = \pi_2(x_1, x_3, x_5, x_{12})$ and express x_{12} as a function of x_1, x_3, x_5 and $\overline{y}_4$. In other words, this ensures that π_{12} represents a valid, positionally dependent, constraint on the yaw angle along the path.

In summary, motivated by the path following problem, we define the "virtual" output function by

$$\overline{y} = \begin{bmatrix} \alpha(x) \\ \pi_1(x_1, x_3, x_5) \\ \pi_2(x_1, x_3, x_5, x_{12}) \end{bmatrix} \tag{6}$$

where $\alpha(x)$ is defined in (5) and π_2 satisfies Assumption 2. While the output function is intuitively appealing for the purposes of meeting **PF1**, **PF2** and **PF3**, the following result shows that it fails to yield a well-defined relative degree.

Lemma 4.1: System (1) with output (6) does not have a well-defined vector relative degree at any $x \in \mathbb{R}^{12}$.

One possible interpretation of Lemma 4.1 is that the decoupling matrix loses rank because the control input u_ϕ does not appear. This problem can be overcome by delaying the appearance of the control input u_c with the help of two integrators. This suggests the inclusion of two controller states $\zeta = (\zeta_1, \zeta_2)$. Let $u_c = \zeta_1$, where ζ_1 is the first controller state. To further delay the appearance of the control input u_c we choose $\zeta_2 = u_1$, where u_1 is the the auxiliary control input. This generates the dynamic control law

$$\begin{aligned} \dot{\zeta}_1 &= \zeta_2 \\ \dot{\zeta}_2 &= u_1 \\ u_c &= \zeta_1 \\ u_\phi &= u_2 \\ u_\theta &= u_3 \\ u_\psi &= u_4. \end{aligned} \tag{7}$$

To simplify notation, we no longer distinguish between the quadrotor's states $(x_1, \cdots, x_{12})$ and the controller states (ζ_1, ζ_2). Let $x_{13} := \zeta_1$ and $x_{14} := \zeta_2$ and let $u := \mathrm{col}(u_1, u_2, u_3, u_4)$. The augmentation of the two controller states to the system is called dynamic extension and the

[1] The dimension $\tilde{q}$ of the controller state ζ is not fixed *a priori*.

extended system is given by

$$\dot{x} = \begin{bmatrix} x_2 \\ R_1 x_{13} \\ x_4 \\ R_2 x_{13} \\ x_6 \\ -g + R_3 x_{13} \\ x_8 x_9 I_1 \\ x_7 x_9 I_2 \\ x_7 x_8 I_3 \\ x_7 + \tan_{11} R_4 \\ x_8 c_{10} - x_9 s_{10} \\ \sec_{11} R_4 \\ x_{14} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & C_x & 0 & 0 \\ 0 & 0 & C_y & 0 \\ 0 & 0 & 0 & C_z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}, \tag{8}$$

With a slight abuse of notation we write the extended model (8) compactly as $\dot{x} = f(x) + \sum_{i=1}^{4} g_i(x) u_i$. We have seen in Lemma 4.1 that one of the problems with with the non-extended model (1) was that the terms multiplying the control input u_ϕ in the decoupling matrix were all identically equal to zero. Through the use of dynamic extension we have delayed the appearance of the input u_c in the second derivatives of α_1, α_2, π_1, which allows all four inputs u_ϕ, u_θ, u_ψ and u_1 to appear in the same derivative. This will help in avoiding the problem of the second column of the decoupling matrix being identically equal to zero. Further analysis of the decoupling matrix for the extended system is still required and will be discussed in Section V. Applying the zero dynamics algorithm to the output (5) and the extended system (8) yields that the path following manifold is given by

$$\Gamma^\star = \Big\{ x \in \mathbb{R}^{14} : \alpha_1(x) = \cdots = \alpha_1^{(4)}(x) = \alpha_2(x) = \cdots = \alpha_2^{(4)}(x) = 0 \Big\}. \tag{9}$$

V. Path Following Controller Design

In this work the path following problem is treated as an instance of the set stabilization problem and the general approach for solving path following problem is applied to a quadrotor [1], [9], [17]. In contrast to the differential flatness based controller which involves finding an output such that the resulting feedback linearized system is fully linear, we have chosen flat outputs that are physically meaningful for the path following manifold. We now refine the definition of π in the virtual output (6) by choosing a specific function. An mapping is introduced that associates to point y in the output space of the quadrotor system, sufficiently close to the path, a number in the domain $\mathbb{D}$ that minimizes the distance from the path γ. This mapping was used in [1] for curves in $\mathbb{R}^2$. Let $\gamma_\epsilon \subset \mathbb{R}^3$ be a tubular neighbourhood of the path γ and define the map

$$\begin{aligned} \varpi :& \gamma_\epsilon \to \mathbb{D} \\ & y \mapsto \arg \inf_{\lambda \in \mathbb{D}} \|y - \sigma(\lambda)\|. \end{aligned} \tag{10}$$

The above function is smooth so long as γ_ϵ is a sufficiently small "tube" around the curve γ. To simplify our exposition we will restrict the class of curves considered so that

$$\frac{\partial \alpha_1}{\partial x_5} = \frac{\partial \alpha_2}{\partial x_1} = \frac{\partial \alpha_2}{\partial x_3} = 0.$$

With these definitions and restrictions, the refined virtual output function is given by

$$\hat{y} = \begin{bmatrix} \alpha_1(x_1, x_3) \\ \alpha_2(x_5) \\ \pi_1(x_1, x_3, x_5) \\ \pi_2(x_1, x_3, x_5, x_{12}) \end{bmatrix} = \begin{bmatrix} s_1 \circ h(x) \\ s_2 \circ h(x) \\ \varpi \circ h(x) \\ \pi_2(x_1, x_3, x_5, x_{12}) \end{bmatrix}, \tag{11}$$

Lemma 5.1: The extended model of the quadrotor (8) with output (11) yields a well-defined vector relative degree of $\{4, 4, 4, 2\}$ at each point on $\Gamma^\star$ where $x_{10} = x_{11} \neq \pm 90^0$ and $\zeta_1 \neq 0$.

Proof: Let $x^\star \in \Gamma^\star$ be arbitrary. By definition of Γ, and since $\Gamma^\star \subseteq \Gamma$, the output $h(x^\star)$ is on the path γ. Let $\lambda^\star \in \mathbb{D}$ be such that $h(x^\star) = \sigma(\lambda^\star)$. By the definition of vector relative degree we must show that

$$L_{g_i} L_f^j \pi_k(x) = L_{g_i} L_f^j \alpha_k(x) \equiv 0$$

for $i \in \{1,2,3,4\}$, $j \in \{0,1,2\}$, $k \in \{1,2\}$ in a neighbourhood of $x^\star$ and that the 4×4 decoupling matrix

$$D(x^\star) = \begin{bmatrix} L_{g_1} L_f^3 \alpha_1(x^\star) & \cdots & L_{g_4} L_f^3 \alpha_1(x^\star) \\ L_{g_1} L_f^3 \alpha_2(x^\star) & \cdots & L_{g_4} L_f^3 \alpha_2(x^\star) \\ L_{g_1} L_f^3 \pi_1(x^\star) & \cdots & L_{g_4} L_f^3 \pi_1(x^\star) \\ L_{g_1} L_f \pi_2(x^\star) & \cdots & L_{g_4} L_f \pi_2(x^\star) \end{bmatrix}, \tag{12}$$

is non-singular. It is easy to check that

$$L_{g_i} L_f^j \pi_1(x) = L_{g_i} \pi_2(x) = L_{g_i} L_f^j \alpha_k(x) = 0$$

for $i \in \{1,2,3,4\}$, $j \in \{0,1,2\}$. To show that the decoupling matrix is non-singular we analyze its determinant. The closed form expressions of the entries of the decoupling matrix are omitted for brevity. The determinant of the decoupling matrix $D(x)$ simplifies to

$$\frac{L^2 \left(\partial_{x_{12}} \pi_2\right) \left(\partial_{x_5} \alpha_2\right) (x_{13})^2 c_{10}}{I_x I_y I_z m^3 c_{11}} \left(\sigma_2' \partial_{x_1} \alpha_1 - \sigma_1' \partial_{x_3} \alpha_1\right). \tag{13}$$

The determinant goes to zero if and only if any term in the numerator of (13) is zero or any term in the denominator is infinity. The terms I_x, I_y, I_z and m are finite constants. The term c_{11} is bounded and is therefore finite; the physical parameter L is non-zero. By Assumption 1 the curve γ is an embedded submanifold which means that at each $y \in \gamma$, ds_y has rank two. Since $dh_x = I$ this shows, using the chain rule, that at each $x^\star \in \Gamma^\star$, $d\alpha_{x^\star}$ must be full rank which imples that $\partial_{x_5} \alpha_2 \neq 0$. By Assumption 2, we have that $\partial_{x_{12}} \pi_2 \neq 0$.

We claim that state x_{13}, which is the first controller state ζ_1 and physically represents u_c, i.e., the combined thrust of all the rotors can not be zero in reasonable flight conditions. Since we desire to follow a path in $\mathbb{R}^3$ we maintain a close to hover condition. Using the same argument as in [1], the term $[\sigma_2' \partial_{x_1} \alpha_1 - \sigma_1' \partial_{x_3} \alpha_1]$ cannot be zero because it is the inner product of two, collinear, non-zero vectors. The vector $\text{col}\,(\partial_{x_1} \alpha_1, \partial_{x_3} \alpha_1)$ represents the gradient vector to the curve at $h(x^\star) = \sigma(\lambda^\star)$ while σ' is the tangent vector to curve at $\sigma(\lambda^\star)$. By definition these vectors are orthogonal. A rotation by 90^0 makes both vectors collinear. Assumption 1 ensures that these vectors are non-zero. Therefore, the inner product of the these two vectors is always non-zero. Thus we have shown that for any $x^\star \in \Gamma^\star$ with roll angle $x_{10} = \phi \neq \pm 90^0$ the $\det(D(x)) \neq 0$, therefore the extended system (8) has a well defined vector relative degree. ∎

It is important to note that $\zeta_1 \neq 0$ is not a restriction because ζ_1 is the augmented state and it can be chosen by the user. It is interesting to note that we also want to avoid the singularity condition where $x_{11} = \theta = \pm 90^0$. This singularity condition can be avoided by choosing another representation of the system. One possible choice could be quaternions instead of Euler angles. Since the extended system (8) has a well defined vector relative degree of $\{4, 4, 4, 2\}$, this implies that the dimension of the zero dynamics is zero $(14 - (4 + 4 + 4 + 2) = 0)$. In other words, we can fully linearize the extended system of the quadrotor. This leads to the definition of a local coordinate transformation.

Corollary 5.2: Let $x^\star \in \Gamma \backslash \{x \in \mathbb{R}^{14} : x_{10} = x_{11} \pm 90^0\}$. There exists a neighbourhood $U \subset \mathbb{R}^{14}$ containing $x^\star$ such that the mapping $T : U \subset \mathbb{R}^{14} \to T(U) \subset \mathbb{R}^{14}$, defined by

$$\begin{bmatrix} \xi_{ji} \\ \eta_{1i} \\ \eta_{2k} \end{bmatrix} = T(x) = \begin{bmatrix} L_f^{i-1} \alpha_j(x) \\ L_f^{i-1} \pi_1(x) \\ L_f^{k-1} \pi_2(x) \end{bmatrix}, \tag{14}$$

for $i \in \{1, 2, 3, 4\}$, $j \in \{1, 2\}$ and $k \in \{1, 2\}$ is a diffeomorphism.

Proof: Let $x^\star \in \Gamma \backslash \{x \in \mathbb{R}^{14} : x_{10} = x_{11} = \pm 90^0\}$. By Lemma 5.1, system (8) with output (11) yields a well-defined vector relative degree of $\{4, 4, 4, 2\}$ at $x^\star$. By [18, Lemma 5.2.1] the row vectors

$$\begin{gathered} d\alpha_1(x^\star), dL_f \alpha_1(x^\star), dL_f^2 \alpha_1(x^\star, dL_f^3 \alpha_1(x^\star) \\ d\alpha_2(x^\star), dL_f \alpha_2(x^\star), dL_f^2 \alpha_2(x^\star, dL_f^3 \alpha_2(x^\star) \\ d\pi_1(x^\star), dL_f \pi_1(x^\star), dL_f^2 \pi_1(x^\star, dL_f^3 \pi_1(x^\star) \\ d\pi_2(x^\star), dL_f \pi_2(x^\star), \end{gathered} \tag{15}$$

are linearly independent. These are the rows of the 14×14 Jacobian matrix $dT_{x^\star}$ which implies that $dT_{x^\star}$ is non-singular. By the inverse function theorem [19, Theorem 5.23] T is a diffeomorphism onto its image. ∎

Using the coordinate transformation T from Corollary 5.2, the system is transformed in (η, ξ) coordinates

$$\begin{aligned} \dot{\xi}_{j1} &= \xi_{j2} \\ \dot{\xi}_{j2} &= \xi_{j3} \\ \dot{\xi}_{j3} &= \xi_{j4} \\ \dot{\xi}_{j4} &= L_f^4 \alpha_j + \sum_{i=1}^{4} L_{g_i} L_f^3 \alpha_j u_i \Big|_{x = T^{-1}(\eta, \xi)} \\ \dot{\eta}_{11} &= \eta_{12} \\ \dot{\eta}_{12} &= \eta_{13} \\ \dot{\eta}_{13} &= \eta_{14} \\ \dot{\eta}_{14} &= L_f^4 \pi_1 + \sum_{i=1}^{4} L_{g_i} L_f^3 \pi_1 u_i \Big|_{x = T^{-1}(\eta, \xi)} \\ \dot{\eta}_{21} &= \eta_{22} \\ \dot{\eta}_{22} &= L_f^2 \pi_2 + \sum_{i=1}^{4} L_{g_i} L_f \pi_2 u_i \Big|_{x = T^{-1}(\eta, \xi)}, \end{aligned} \tag{16}$$

for $j \in \{1, 2\}$. This coordinate transformation suggest a natural choice of feedback transformation

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} := D^{-1}(x) \left(\begin{bmatrix} -L_f^4 \alpha_1 \\ -L_f^4 \alpha_2 \\ -L_f^4 \pi_1 \\ -L_f^3 \pi_2 \end{bmatrix} + \begin{bmatrix} v^{\xi_1} \\ v^{\xi_2} \\ v^{\eta_1} \\ v^{\eta_2} \end{bmatrix} \right), \tag{17}$$

where $(v^{\xi_1}, v^{\xi_2}, v^{\eta_1}, v^{\eta_2})$ are auxiliary control inputs. By Lemma 5.1 this controller (17) is well-defined in a neighbourhood of every $x^\star \in \Gamma \backslash \{x \in \mathbb{R}^{14} : x_{10} = x_{11} = \pm 90^0\}$. Thus in a neighbourhood of $x^\star$, the closed loop (8) is simply reduced to 4 decoupled chains of integrators.

$$\begin{array}{llll} \dot{\xi_{11}} = \xi_{12} & \dot{\xi_{21}} = \xi_{12} & \dot{\eta_{11}} = \eta_{12} & \dot{\eta_{21}} = \eta_{12} \\ \dot{\xi_{12}} = \xi_{13} & \dot{\xi_{22}} = \xi_{13} & \dot{\eta_{12}} = \eta_{13} & \dot{\eta_{22}} = v^{\eta_2}. \\ \vdots & \vdots & \vdots & \\ \dot{\xi_{14}} = v^{\xi_1} & \dot{\xi_{24}} = v^{\xi_2} & \dot{\eta_{24}} = v^{\eta_2} & \end{array} \tag{18}$$

The above system consists of 4 decoupled linear time invariant (LTI) systems and any linear control technique can be used to stabilize (18). We call the first chain of integrators ξ_1-subsystem, the second chain of integrators ξ_2-subsystem, the third chain of integrators η_1-subsystem and fourth chain of integrators η_2-subsystem. The output (11) is a flat output [20] for the quadrotor system (1) because these outputs transform the system to a fully linear system. The linear form of the quadrotor in the transformed coordinates simplifies the design of path following controllers.

A. Auxiliary controller design

After applying the coordinate and feedback transformations (14), (17) to the extended system (8) the auxiliary controller design is straight forward. Stabilizing the origin of the first two chain of integrators, collectively called the ξ-subsystem, corresponds to the stabilization of path following manifold $\Gamma^\star$. When $\xi = 0$ the states of the system are restricted to stay on the path. We propose the following controller to control the ξ-subsystem.

$$v^{\xi_j} = \sum_{i=1}^{4} k_i \xi_{ji}, \tag{19}$$

with $k_i < 0$, $j \in \{1, 2\}$. This controller exponentially stabilizes $\xi = 0$. Since $\xi = 0$ is an equilibrium of the ξ-subsystem, the origin is exponentially stable. Moreover, we stabilize the path following manifold $\Gamma^\star$ and hence path invariance is achieved. In other words, **PF1** and **PF2** are satisfied.

To achieve the goal of point stabilization along the curve, controlling the speed along the curve, and forcing the quadrotor to follow a given acceleration profile along the curve we propose the following controller

$$v^{\eta_1} = \sum_{i=1}^{4} k_i(\eta_{1i} - \eta_{1i}^{ref}), \tag{20}$$

where $k_i < 0$, η_{11} is the path parameter. By setting η_{11}^{ref} to the desired value, point stabilization is achieved. By choosing $k_1 = 0$ and setting η_{12}^{ref} to the desired velocity profile the quadrotor follows the given velocity profile. Similarly, by choosing $k_1 = k_2 = 0$ and setting η_{13}^{ref} to the desired acceleration profile the quadrotor follows the given acceleration profile. Therefore, by the auxiliary controller given by (20), **PF3** has been achieved.

The yaw angle of the quadrotor can be controlled by designing a similar controller for the fourth chain of integrator.

$$v^{\eta_2} = k_1(\eta_{21} - \eta_{21}^{ref}) + k_2\eta_{22}, \tag{21}$$

where $k_1, k_2 < 0$. By stabilizing the origin of the η_2-subsystem, the yaw angle of the quadrotor converges to the desired yaw angle reference function, satisfying the yaw objective in **PF3**.

VI. Simulation Results

In this section, we present simulation results of the quadrotor with the dynamic path following controller designed in section V. For simulation purposes, it is assumed that the quadrotor has a mass of $m = 1$ kg, and inertias $I_x = I_y = I_z = 1$ N.m/rad/sec^2, a length $L = 0.3$ m and acceleration due to gravity is $g = 9.8$ m/sec^2. It is further assumed that due to modeling uncertainties there is 10% error in I_x, I_y, I_z and L. The error in the mass of the quadrotor is assumed to be 1% because the mass of the quadrotor can be accurately measured by a precise weight measuring instrument. The initial position of the quadrotor is indicated by a solid dot. The quadrotor is following a curve represented by a fourth order spline. The controller allows the quadrotor to follow a spline of any order greater than 1. Moreover, the quadrotor is capable of following any closed or non-closed curve satisfying Definition 1. The path chosen for this simulation is a general 4$^{\text{th}}$ order spline given by $\sigma : \mathbb{R} \to \mathbb{R}^3$, $\lambda \mapsto \text{col}(\lambda, a_4\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0, 3)$. The implicit representation of the same curve is given by $\gamma = \{s_1(y) = s_2(y) = 0\}$, where $s_1(y) = x_3 - a_4x_1^4 + a_3x_1^3 + a_2x_1^2 + a_1x_1 + a_0 = 0$ and $s_2 = x_3 - 3 = 0$. In Figure 2, the quadrotor is following the desired path and following a velocity profile

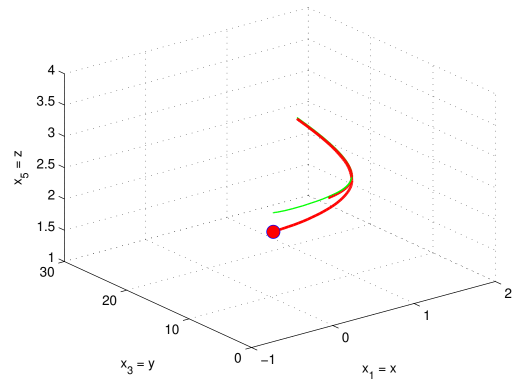

Fig. 2. Velocity profile simulation. The path followed by the quadrotor is represented by a bold line and the desired path is represented by a dashed line.

$$\eta_{12}^{ref} = \begin{cases} 1 & 0 \le t < 20\text{s} \\ 0 & 20 \le t < 40\text{s} \\ -1 & t \ge 40\text{s}. \end{cases}$$

The interesting feature about the above mentioned velocity profile is that it forces the quadrotor to maintain a velocity of 1 unit/sec for 20 sec, stopping along the curve for the next 20 sec, and then reverse the direction of the motion with a velocity of -1 unit/sec. This indicates that with the proposed controller, the quadrotor is capable of stopping along the curve and changing direction along the curve while staying on the path. A quadrotor velocity is compared to the desired velocity in Figure 3. Moreover, the quadrotor is following

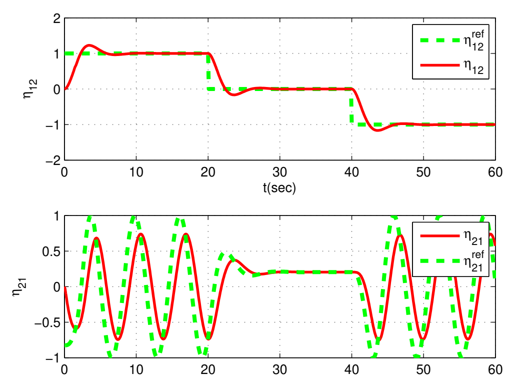

Fig. 3. Comparison between reference and actual states.

a yaw profile $\eta_{21}^{ref} = \sin(\eta_{12})$ while following the desired path as shown in Figure 3. The interesting feature about this profile is that it does not depended on time but on the path parameter λ, which is equal to η_{12} in this case. Stabilizing the yaw angle to a desired value or forcing it to follow a given profile can be of practical importance as quadrotors are often used for aerial surveillance and inspection tasks with rigidly mounted cameras. The yaw stabilization would allow the camera to be pointed in a particular direction or along a particular desired yaw profile throughout the flight along the desired path. The simulation results show that with the proposed controller the quadrotor is capable of following a large class of closed and non-closed curves with reasonable modeling uncertainties in the system's parameters.

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